ELEMENTS

OF

WAVE MOTION

RELATING TO

SOUND AND LIGHT.

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A TEXT BOOK

PREPARED EXPRESSLY FOR THE USE OF THE

CADETS OF THE UNITED STATES MILITARY ACADEMY,

WEST POINT.

BY

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PREFACE.

THIS text-book, as is stated on the title-page, has been prepared expressly for the use of the Cadets of the United States Military Academy, and this specific object has therefore wholly controlled its design and restricted its scope. It is thus in no sense a treatise. Because of the limited time allotted to the subjects of sound and light in the present distribution of studies at the Academy, the problem of arranging a fundamental course of sufficient strength, to be something more than popular, and yet to be mastered within the allotted time, has been somewhat perplexing. The basis of this arrangement is necessarily the mathematical attainments of the class for which the course is intended. In this respect, the class has completed the study of elementary Mathematics, as far as to include the Calculus, and has had a four months' study of the application of pure Mathematics, in a course of Analytical Mechanics. With these elements to govern, this text-book has been designed for a seven weeks' course, including advance and review. The fact of being able, through the discipline of the Academy, to exact of each student a certain number of hours of hard study on each lesson, is of course an important element necessary to be stated.

The study of the text is supplemented by lectures, in which the principles of Acoustics and Optics are amply illustrated by the aid of a very well equipped laboratory of physical apparatus. Carefully written notes of the lectures are submitted by each student to the instructor on the following morning for revision and criticism. Important errors of fact and misinterpretation of principle are thus at once detected, corrected, and hence prevented from obtaining a lodgment in the mind of the pupil. Another element in this matter of instruction, of sufficient importance to be mentioned, is the opportunity freely exercised by each student of making known the difficulties that he has encountered, before being called upon to exhibit his proficiency in the lesson of the day. It is required that these difficulties shall be clearly and exactly stated, in order that the instructor may, by a judicious question or a concise explanation, enable the student to clear up the difficulty as of himself, and thus complete the elucidation.

The author believes that this method of instruction, taken as a whole, is in sufficiently intimate accord with the text as to gain the following advantages, viz.: 1°, the tasks are of the requisite strength to demand all the study-time allotted to his department of instruction, and thus is secured the invaluable mental effort and discipline due to a specified number of hours of hard study; 2°, while the daily tasks are progressive, they are based on fundamental principles which require the exercise of a rational faith, and develop a continual growth of confidence in the mind of the pupil, and a belief in his own ability to overcome each difficulty as it arises; 3°, when the course is completed, the student finds himself equipped with a satisfactory knowledge of the essential principles of the physical science, to which he may add by further individual study, without the necessity of reconstructing his foundation.

The elements of character developed in the student by this system of instruction, viz., confidence in his powers, reliance on individual effort, and capacity to appreciate truly his sources of information, are of essential importance in a career where he may be called upon in emergencies to exercise self-control, and to meet manfully unforeseen difficulties; and they offer a sufficient reason for the importance given to these studies in the curriculum of the Academy.

Text-books are generally compilations. The subject-matter of this text has been gathered by the author from whatever source appeared to him best for the purpose in view. And as it is often desirable to refer to original treatises, for a better conception of the subject under discussion, a list of authors is appended to this Preface.

In the arrangement of the matter, the author has been governed alone by the necessities of the case and the restrictions of the course. It has therefore seemed advisable to arrive at the deduction of Fresnel's wave surface as expeditiously as possible, and on the way to establish all of the essential principles of undulatory motion common to sound and light. Sufficient theoretical attention is paid in the text to the wave surface, and a study of its model in the lecture-room makes clear its important properties and those of its special cases. Acoustics is briefly treated, and is indeed made subsidiary to Optics, by utilizing its numerous illustrations in vibratory motion, so that the laws of this motion may be the more clearly apprehended in the subject of light. In Optics, while the essential principles of the deviation of light by lenses and mirrors, the construction of optical images and the principal telescopic combinations, are carried only to first approximations, and are somewhat more condensed than is usual, nothing essential to the Academic course of Astronomy has been omitted. The part relating to physical Optics is very concise, but the experiments performed and illustrations given in the lecture room, especially in diffraction, dispersion, and polarization, largely remedy this defect.

The figures throughout the text were drawn by Lieut. Arthur Murray, 1st U. S. Artillery, Acting Asst. Professor of Philosophy, U. S. M. A., to whom I desire to acknowledge my great indebtedness.

P. S. M.

WEST POINT, N. Y., May, 1882.

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PART I.

WAVE MOTION.

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1. Equation (E) of Analytical Mechanics (Michie),

$$\Sigma I\delta p - \Sigma m \frac{d^2s}{dt^2}\delta s = 0,$$

expresses in mathematical language the law that the potential energy expended is equal to the kinetic energy developed. Every analytical discussion of the action of force upon matter must be founded upon this general equation. For the complete solution of every problem of energy, it is necessary to know the intensities, lines of action, and points of application of the acting forces, the masses acted upon, and to possess a perfect mastery of such mathematical processes as are necessary to pass to the final equations whose interpretation will make known the effects. These difficulties, which, in Mechanics, limit the discussion to the free, rigid solid, and to the perfect fluid, are, in Molecular Mechanics, almost insuperable; since we neither know the nature of the forces which unite the elements of a body into a system, nor the constitution of the elements themselves.

2. But the faculty of observation, being cultivated and logically directed, has enabled scientific men to originate experiments which, because of our inherent faith in the uniformity of the laws of nature, have resulted in certain hypotheses as to the nature of sound, light, heat, and other molecular sciences. When an hypothesis not only satisfactorily explains the known phenomena of the science in question, but even predicts others, it then becomes a theory, and its acceptance is more or less complete. An hypothesis is related to a theory as the scaffolding to the structure, the latter being so proportioned in all its parts as to be in the completest harmony, while the former may be modified in any way to suit the ever-varying necessities of the architect.

While there are many matters concerning which a reasonable doubt may be entertained, because of insufficient data, the progress of scientific thought and the fertility of scientific research have, within recent times, establ shed certain facts that are now universally accepted.

3. ***Molecular Science.*** Molecular science is a branch of Mechanics in which the forces considered are the attractions and repulsions existing among the molecules of a body, and the masses acted upon are the indefinitely small elements, called molecules, of which the body is composed. It embraces light, heat, sound, electrics, and, in one sense, chemistry.

4. From the facts of observation and experiment, it is assumed that all matter, whether solid, liquid, or gaseous, is made up of an innumerable number of molecules in sensible, though not in actual contact; that these molecules are so small as not to be within range of even our assisted vision; and that they are separated from each other by distances which are very great compared with their actual linear dimensions.

5. The molecular forces, which determine the particular state of the matter, are either attractive or repulsive. When the attractive forces exceed the repulsive in intensity, the body is a solid; when equal to the repulsive, a liquid; and when less, a gas. The relative places of equilibrium of the molecules are determined by the molecular forces called into play by the action of extraneous forces applied to the body. Thus, when a solid bar is subjected to the action of an extraneous force, either to elongate or to compress it, the molecules assume new positions of equilibrium with each increment of force, and, in either case, the aggregate molecular forces developed are equal in intensity, but contrary in direction, to the extraneous force applied. In general, where rupture does not ensue, the extraneous forces applied are much less than the molecular forces capable of being called into play.

6. While we are ignorant of the true nature of force and matter, our senses enable us to appreciate the effects of the former upon the latter. Our whole knowledge of the physical sciences is based upon the correct interpretation of these sensuous impressions. Observation teaches that if a body be subjected to the action of an extraneous force, the effect of the force is transmitted throughout

the body in all directions, and since the body is connected with the rest of the material universe, there is no theoretical limit to the ultimate transfer of this effect throughout space.

7. Among the appreciable effects of force are the changes of state with respect to rest and motion. These can be transferred from an origin to another point in but two ways, viz.:

1°. By the simultaneous transfer of the body, which is the depository of the motion.

2°. By the successive actions and reactions between the consecutive molecules along any line from the origin.

In the molecular sciences, the latter is assumed to be the method of transfer, and the object of the succeeding discussion is to investigate the nature of the disturbance, the circumstances of its progress, and the behavior of the molecules as they become involved in it.

8. While the initial disturbance is perfectly arbitrary, the molecular motions produced through its influence in any medium are necessarily subjected to the variable conditions which result from the action of the forces that unite the molecules into a material system. The problems are then those of constrained motion.

9. Among the physical properties of bodies, elasticity is of such great importance, that a complete knowledge of its mathematical theory is essential to the thorough elucidation of many of the phenomena of molecular science. The limits of this text permit but a passing allusion to its more important laws.

ELASTICITY.

10. A body is said to be homogeneous when it is formed of similar molecules, either simple or compound, occupying equal spaces, and having the same physical properties and chemical composition. In such a body, a right line of given length l and determinate direction is understood to pass through the same number n of molecules wherever it is placed; the ratio $\frac{l}{n}$ will vary with the direction of l. In crystalline bodies, considered as homogeneous, $\frac{l}{n}$ varies with the direction; in homogeneous non-crystalline bodies, such as glass, the ratio varies insensibly, or is independent of the direction. This supposition requires n to be very great, however small l may be.

11. That property, by which the internal forces of a body or medium restore, or tend to restore, the molecules to their primitive positions, when they have been moved from these positions by the action of some external force, is called *Elasticity*.

12. The elasticity is said to be perfect when the body always requires the same force to keep it at rest in the same bulk, shape, and temperature, through whatever variations of bulk, shape, and temperature it may have been subjected.

13. Every body has some degree of elasticity of bulk. If a body possess any degree of elasticity of shape, it is called a solid; if none, a fluid. All fluids possess great elasticity of bulk. While the elasticity of shape is very great for many solids, it is not perfect for any. The degree of distortion within which elasticity of shape is found, is essentially limited in every solid; when the distortion is too great, the body either breaks or receives a permanent *set;* that is, such a molecular displacement that it does not return to its original figure when the distorting force is removed.

14. The limits of elasticity of metal, stone, crystal, and wood are so narrow that the distance between any two neighboring molecules of the substance never alters by more than a small proportion of its own amount, without the substance either breaking or experiencing a permanent set. In liquids, there are no limits of elasticity as regards the magnitude of the positive pressures applied; and in gases, the limits of elasticity are enormously wider with respect to rarefaction than in either solids or liquids, while there is a definite limit in condensation when the gas is near the critical temperature.

15. The substance of a homogeneous solid is called *isotropic* when a spherical portion exhibits no difference, in any direction, in quality, when tested by any physical agency. When any difference is thus manifested, it is said to be *æolotropic.*

16. ***Origin of the Theory of Elasticity.*** In Mechanics, by supposing the bodies perfectly rigid, and the distances of the points of application of the extraneous forces invariable, however great the forces, the problems are much simplified, without affecting their generality. But this ignores the law by which the reciprocal influence is transmitted from point to point of the body, and by which the action of one force is counterbalanced by the actions of others. In reality, the body undergoes deformation, and when the limit is reached, rupture ensues. The mathematical theory has arisen from the necessity of a knowledge whereby these permanent deformations and rupture may be avoided. This theory has been extended to the determination of the laws of small motions, or, in general, to the vibrations of elastic media.

17. The initial state of a homogeneous body is considered to be that in which it is perfectly free from all extraneous forces, to be, indeed, that of a body falling freely in vacuo. Such a body is then the geometrical place of an innumerable number of material points, which are distinguished from the rest of space by several mechanical properties. Each of these material points is called a molecule.

18. When such a body is subjected to the action of an extraneous force, either a tension or a pressure, a motion of its surface particles ensues, and this disturbance is propagated to the interior molecules; the body becomes slightly distorted, and soon takes a new state of equilibrium. When the external forces are removed, the internal forces are again balanced, and the original condition is restored, provided there is no permanent set. All changes of form of a solid, or any variation of the relative distances of its material points, are ever accompanied by the development of attractive or repulsive forces between the molecules. These variations and forces begin, increase, decrease, and end at the same time, and hence are mutually dependent.

19. The properties of a solid body depending only upon those of its material points, they alone are the foci whence emanate these interior forces.

20. Let an extraneous force be applied to a body, and consider its effect upon any two molecules sufficiently near each other to be mutually affected by their changes of position. Should one of the molecules, on account of this exterior action, approach the other, a mutual repulsion takes place, which, in time, overcomes the motion of the first molecule, and causes the second to take its new position of equilibrium with respect to the first. The reverse is the case when the first molecule withdraws from the second, and an attractive force is developed between them. If r represent the primitive distance, Δr may represent the displacement. Then the intensity of the attractive or repulsive force developed between the molecules may be represented by $f(r, \Delta r)$. This function becomes zero when Δr is zero, whatever r may be; it decreases rapidly when r has a sensible value, whatever Δr may be, since all cohesion ceases between two parts of the same body separated by an appreciable distance. Assuming that the intensity of the molecular forces varies directly with the degree of displacement, this limitation embodies only the cases where the changes of form are very small, whether the extraneous forces are extremely small or the bodies considered have great rigidity. Hence, $f(r, \Delta r)$ is limited to the product of a function of r and the first power of Δr, which becomes infinitely small when Δr becomes infinitely small.

21. *Elastic Force defined.* From any molecule M in the interior of a solid, with a radius equal to the greatest distance beyond which $f(r)$ is insensible, describe a sphere. This volume will embrace all molecules that influence the molecule M, and may be called the *sphere of molecular activity*. Pass a plane through M, dividing the sphere into the two parts SAC and SBC. Normal to LN and having for its base a differential surface ω, conceive a cylinder in the hemisphere SBC. When the equilibrium is disturbed, the molecules in SAC will act on the molecules of the cylinder. The resultant ωE of all these actions is called the *elastic force* exerted by SAC upon SBC, referred to the infinitesimal surface ω. Integrating this function with respect to the plane, we obtain the elastic force referred to the circle SMC. The resultant ωE will, in general, be oblique to the plane element ω. If it is normal to this element and directed towards the hemisphere SAC, it will be a traction; if normal and directed toward SBC, it will be a pressure; if parallel to the plane SMC, it will be the tangential elastic force.

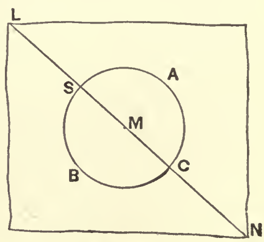


Figure I.

Similarly, if the cylinder is situated in the hemisphere SAC, the resultant elastic force exerted upon the molecules of the cylinder by the molecules in SBC is represented by $\omega E'$, referred to the same elementary surface ω. If the body, slightly changed in form, is in equilibrium of elasticity, the two elastic forces ωE and $\omega E'$ should be equal in intensity, but contrary in direction. Both, however, will represent either pulls, pressures, or tangential forces; that is, if one is a pull, the other will be a pull directly opposed to it.

The elastic force ωE, considered with reference to the element planes ω drawn parallel to each other through all points of the body, will vary in intensity and direction from point to point; and at the same point M will vary with the orientation of the element plane ω.

22. The direction of the planes ω may be determined by that of their normals. Using the angles ϕ and ψ to designate the latitude and longitude of the point where the normal pierces the surface of the sphere of activity, and representing by x, y, and z the co-ordinates of this point referred to the co-ordinate axes, we have

$$x = \cos\phi \cos\psi, \qquad y = \cos\phi \sin\psi, \qquad z = \sin\phi.$$

Representing the orthographic projections of ωE by ωX, ωY, and ωZ upon the co-ordinate axes, we see, in the case of equilibrium of elasticity, that ωE will be a function of the five variables x, y, z, ϕ, and ψ; and if the motion be progressive, the variable t will also enter. X, Y, and Z can be determined from ωE, ϕ, and ψ; and, reciprocally, the latter from the former. X, Y, and Z are, however, usually determined, and are, in general, functions of the six variables (x, y, z, ϕ, ψ, t), and which being found according to the special circumstances that cause the deformation of the body, would enable us to ascertain, at each instant and at each point of the body, the direction and intensity of the elastic force exerted upon every element plane passing through the given point. In brief, the determination of these functions and the study of their properties are the principal objects of the mathematical theory of elasticity.

23. ***Elasticity of Solids.*** Experiment has shown that, when a solid bar is subjected to small elongations, or those within elastic limits, the following laws are verified, viz.: 1°, the elongations are directly proportional to the length of the bar; 2°, they are inversely proportional to the area of cross section; 3°, they are directly proportional to the intensity of the elongating force; 4°, they are variable for bars of different materials. These experimental laws can be expressed by the equation,

$$\lambda = \frac{1}{M} \cdot \frac{Pl}{s}, \qquad (1)$$

in which l is the length of the bar unloaded, s the area of cross-section, P the intensity of the stretching force, M a coefficient varying with the nature of the material, and λ is the corresponding elongation. Making $s = 1$, $\lambda = l$, we get, from the above equation, $P = M$. If, therefore, the law of the elongation should remain true for all intensities, M would be that intensity which, applied to a bar of unit area in cross-section, would make the elongation equal to the original length. Such an hypothesis gives us the value of the coefficient M, which can be used within the limits of experiment. M is called the *coefficient* or *modulus* of longitudinal elasticity, or Young's modulus. While we cannot experiment over such wide limits in longitudinal compression, because of the liability to flexure, the same laws are held to be applicable, with the same limitations. Taking the metre for the unit of length, the square centimetre for the unit of area, and the gramme for the unit of intensity, the moduli of longitudinal elasticity for the principal metals are, according to Wertheim, as follows:

Lead, ...	177×10^6	Copper, ...	1245×10^6
Gold, ...	813×10^6	Platinum, ...	1704×10^6
Silver, ...	736×10^6	Iron, ...	1861×10^6
Zinc, ...	873×10^6	Steel, ...	1955×10^6

The coefficient of elasticity decreases with increase of temperature between 15° and 200° C.

24. An isotropic solid has, in addition to the modulus of longitudinal elasticity, a modulus of rigidity; the former relating to the elasticity of bulk or volume, and the latter to that of shape. If a bar be of square cross-section before elongation, it will be found afterwards to have undergone deformation in its angles, although the diagonals of the cross-section may still be at right angles. The numerical ratio of the intensity of the force applied, to the deformation produced is the *modulus of rigidity*. The deformation is measured by the change in each of the four right angles, in terms of the radian (57°.29) as unity.

25. *Fundamental Coefficients of Elasticity.* Let there be a rectangular parallelopipedon AH, subjected at first to the action of equal and opposite normal pressures on the two bases AD and EH. The vertical edges will, by the laws of elongation, shorten, and the horizontal edges increase in length; and the relative changes in length will be proportional to the quotient of the normal pressures by the area AD; that is, to the pressure on the unit of area.

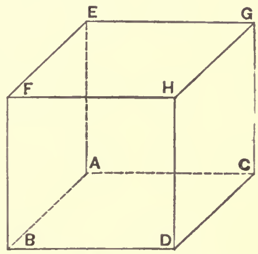


Figure 2.

Let α be the relative shortening of the vertical edges, β the relative increase of the horizontal edges, and P the pressure on the unit of area, then

$$\alpha = mP, \qquad \beta = nP,$$

m and n being coefficients to be determined only by experiment. If Q be the pressure applied to the unit area on the faces AF and CH, the edge AC will be shortened α', and the edges AB, AE lengthened β', and we will have

$$\alpha' = mQ, \qquad \beta' = nQ.$$

If R be the pressure on the unit area of the faces AG and BH, the edge AB will be shortened α'', and the edges AE and AC elongated β'', and we will have

$$\alpha'' = mR, \qquad \beta'' = nR.$$

If now the three pairs of pressure, P, Q, R, act simultaneously, their effects will be superposed, and, representing by ε, ε', ε'', the relative variations of the lengths of the edges AE, AC, and AB, we will have

$$\left.\begin{aligned} \varepsilon &= \alpha - (\beta' + \beta'') = mP - n(Q + R), \\ \varepsilon' &= \alpha' - (\beta + \beta'') = mQ - n(P + R), \\ \varepsilon'' &= \alpha'' - (\beta + \beta') = mR - n(P + Q); \end{aligned}\right\} \qquad (2)$$

from which we readily deduce,

$$\left.\begin{aligned} P &= H\varepsilon + K(\varepsilon' + \varepsilon''), \\ Q &= H\varepsilon' + K(\varepsilon + \varepsilon''), \\ R &= H\varepsilon'' + K(\varepsilon + \varepsilon'); \end{aligned}\right\} \qquad (3)$$

in which

$$\left.\begin{aligned} H &= \frac{m-n}{m(m-n)-2n^2}, \\ K &= \frac{n}{m(m-n)-2n^2}. \end{aligned}\right\} \qquad (4)$$

Hence the pressures exerted upon the faces of the volume, and therefore the elastic reactions, can be expressed as linear functions of the relative variations of the length of the edges by means of two constant coefficients. These two coefficients, H and K, are fundamental in the theory of elasticity. They can only be determined by experimental investigations; once determined for any body, the problems of elasticity become those of rational mechanics.

Exact analysis of the conditions of equilibrium in the interior of a solid elastic body shows that, in each point of the body, there exist three rectangular directions, variable from one point to another, such that the elements perpendicular to these directions support normal pressures or tractions.

An infinitely small parallelopipedon, having its edges parallel to these three directions, is in the condition of that discussed above; and it suffices to express, in a general manner, the relations which exist between the pressures which it sustains and the changes of length of its infinitely small dimensions, to obtain the differential equations of the problem under consideration.

26. Equations (3) can be written,

$$\left.\begin{aligned} P &= (H-K)\varepsilon + K(\varepsilon + \varepsilon' + \varepsilon''), \\ Q &= (H-K)\varepsilon' + K(\varepsilon + \varepsilon' + \varepsilon''), \\ R &= (H-K)\varepsilon'' + K(\varepsilon + \varepsilon' + \varepsilon''). \end{aligned}\right\} \qquad (5)$$

Calling θ the relative variation of the volume, or cubic dilatation, we may, because of the small values of the deformations, write

$$\theta = \varepsilon + \varepsilon' + \varepsilon''. \qquad (6)$$

Placing $H - K = 2\mu$ and $K = \lambda$, we have

$$\left.\begin{aligned} P &= \lambda\theta + 2\mu\varepsilon, \\ Q &= \lambda\theta + 2\mu\varepsilon', \\ R &= \lambda\theta + 2\mu\varepsilon''. \end{aligned}\right\} \qquad (7)$$

Each of the tractions or pressures is then the sum of a term proportional to the cubic dilatation and of a term proportional to the linear dilatation parallel to the pressure considered.

27. A liquid parallelopipedon can be in equilibrium only when the pressures exerted on its six faces are equal; and we know besides that the increase of density or negative increase of volume of the liquid is proportional to the pressure. We will then have

$$P = Q = R = \lambda\theta. \tag{8}$$

The same general theory thus comprises both liquids and solids, in admitting the coefficient 2μ of the former to be zero. The variation of this coefficient from zero marks the departure of the body from the perfect liquid state and its approach to that of the solid.

28. *Analytical expression of the elastic forces developed in the motion of a system of molecules, solicited by the forces of attraction or repulsion, and subjected to small displacements from their positions of equilibrium.*

Let x, y, z, and $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, be the rectangular coordinates of the two molecules of the system, whose masses are respectively m and μ, and whose distance apart is r. The intensity of the reciprocal action of the molecules, being exerted along the right line joining them, is

$$m\mu\, f(r),$$

$f(r)$ being an undetermined function of the distance. If the system is in equilibrium, we have the relations,

$$\left.\begin{aligned} m\,\Sigma\mu\, f(r)\frac{\Delta x}{r} &= 0, \\ m\,\Sigma\mu\, f(r)\frac{\Delta y}{r} &= 0, \\ m\,\Sigma\mu\, f(r)\frac{\Delta z}{r} &= 0. \end{aligned}\right\} \tag{9}$$

At a certain instant, let us suppose that the molecules of the system are displaced from their positions of equilibrium by a very small distance, and let ξ, η, ζ, be the projections of the displacement ε of the molecule m on the axes; let $\xi+\Delta\xi$, $\eta+\Delta\eta$, $\zeta+\Delta\zeta$, be the projections of the displacement of the molecule μ on the axes; and $r+\rho$ the new distance between the molecules. Representing the components of the elastic force parallel to the axes exerted upon the molecule m by all the molecules μ within the sphere of molecular activity, by $X\varepsilon$, $Y\varepsilon$, $Z\varepsilon$, so that X, Y, Z, are the components of the elastic force for a displacement *unity* in the same directions, we have

$$\left.\begin{aligned} X\varepsilon &= m\,\Sigma\mu\,f(r+\rho)\,\frac{\Delta x + \Delta\xi}{r+\rho}, \\ Y\varepsilon &= m\,\Sigma\mu\,f(r+\rho)\,\frac{\Delta y + \Delta\eta}{r+\rho}, \\ Z\varepsilon &= m\,\Sigma\mu\,f(r+\rho)\,\frac{\Delta z + \Delta\zeta}{r+\rho}. \end{aligned}\right\} \quad (10)$$

29. Developing $f(r+\rho)$, and neglecting the terms of a higher order than those containing ρ, since the displacements are regarded as very small, we obtain, recollecting that $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, are of the same order of magnitude as ρ, while Δx, Δy, Δz, may be of any order whatever,

$$\left.\begin{aligned} X\varepsilon &= m\,\Sigma\mu\,[f(r) + \rho f'(r)]\left(\frac{\Delta x + \Delta\xi}{r}\right)\left(1 - \frac{\rho}{r}\right), \\ &= m\,\Sigma\mu\left[f(r)\frac{\Delta\xi}{r} + \rho f'(r)\frac{\Delta x}{r} - \rho f(r)\frac{\Delta x}{r^2}\right]; \\ &= m\,\Sigma\mu\left\{f(r)\frac{\Delta\xi}{r} + \left[f'(r) - \frac{f(r)}{r}\right]\rho\frac{\Delta x}{r}\right\}; \end{aligned}\right\} \quad (11)$$

we also have $$r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2, \quad (12)$$

$$(r+\rho)^2 = (\Delta x + \Delta\xi)^2 + (\Delta y + \Delta\eta)^2 + (\Delta z + \Delta\zeta)^2; \quad (13)$$

from which $$\rho = \frac{\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta}{r}. \quad (14)$$

Substituting this value of ρ in equations (11), we obtain

$$\left.\begin{aligned} X\varepsilon = m\,\Sigma\mu\Bigg\{&\left(\frac{f(r)}{r} + \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x^2}{r^2}\right)\Delta\xi \\ &+ \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x\,\Delta y}{r^2}\,\Delta\eta \\ &+ \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x\,\Delta z}{r^2}\,\Delta\zeta\Bigg\}. \end{aligned}\right\} \quad (15)$$

Similarly, for the axes Y and Z we get,

$$\left.\begin{aligned} Y\varepsilon = m\,\Sigma\mu\Big\{&\left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x\,\Delta y}{r^2}\,\Delta\xi \\ &+ \left(\frac{f(r)}{r} + \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta y^2}{r^2}\right)\Delta\eta \\ &+ \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta y\,\Delta z}{r^2}\,\Delta\zeta\Big\}, \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} Z\varepsilon = m\,\Sigma\mu\Big\{&\left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x\,\Delta z}{r^2}\,\Delta\xi \\ &+ \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta y\,\Delta z}{r^2}\,\Delta\eta \\ &+ \left(\frac{f(r)}{r} + \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta z^2}{r^2}\right)\Delta\zeta\Big\}. \end{aligned}\right\} \quad (17)$$

Putting $\phi(r)$ for $\dfrac{f(r)}{r}$ $\psi(r)$ for $f'(r) - \dfrac{f(r)}{r}$; and $m\dfrac{d^2\xi}{dt^2}$, $m\dfrac{d^2\eta}{dt^2}$, $m\dfrac{d^2\zeta}{dt^2}$, for their equals $X\varepsilon$, $Y\varepsilon$, $Z\varepsilon$, we have

$$\left.\begin{aligned} X\varepsilon = m\frac{d^2\xi}{dt^2} = m\,\Sigma\mu\Big\{&\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\Delta\xi \\ &+ \psi(r)\frac{\Delta x\,\Delta y}{r^2}\Delta\eta + \psi(r)\frac{\Delta x\,\Delta z}{r^2}\Delta\zeta\Big\}, \\ Y\varepsilon = m\frac{d^2\eta}{dt^2} = m\,\Sigma\mu\Big\{&\psi(r)\frac{\Delta x\,\Delta y}{r^2}\Delta\xi \\ &+ \left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta\eta + \psi(r)\frac{\Delta y\,\Delta z}{r^2}\Delta\zeta\Big\}, \\ Z\varepsilon = m\frac{d^2\zeta}{dt^2} = m\,\Sigma\mu\Big\{&\psi(r)\frac{\Delta x\,\Delta z}{r^2}\Delta\xi \\ &+ \psi(r)\frac{\Delta y\,\Delta z}{r^2}\Delta\eta + \left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right]\Delta\zeta\Big\}; \end{aligned}\right\} \quad (18)$$

which give the values of the component elastic forces developed in any molecule of the medium, when the displacements are small.

30. If the displacement is only in the direction of each axis in succession, we have the following groups of equations.

Of x**:**

$$\left.\begin{aligned} X_1 &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\Delta\xi, \\ Y_1 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta y}{r^2}\right]\Delta\xi, \\ Z_1 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta z}{r^2}\right]\Delta\xi; \end{aligned}\right\} \qquad (19)$$

Of y**:**

$$\left.\begin{aligned} X_2 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta y}{r^2}\right]\Delta\eta, \\ Y_2 &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta\eta, \\ Z_2 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta y\,\Delta z}{r^2}\right]\Delta\eta; \end{aligned}\right\} \qquad (20)$$

Of z**:**

$$\left.\begin{aligned} X_3 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta z}{r^2}\right]\Delta\zeta, \\ Y_3 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta y\,\Delta z}{r^2}\right]\Delta\zeta, \\ Z_3 &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right]\Delta\zeta. \end{aligned}\right\} \qquad (21)$$

31. Combining the above equations, we have

$$\left.\begin{aligned} X\varepsilon &= X_1 + X_2 + X_3, \\ Y\varepsilon &= Y_1 + Y_2 + Y_3, \\ Z\varepsilon &= Z_1 + Z_2 + Z_3. \end{aligned}\right\} \qquad (22)$$

From Eqs. (19) we see that the total intensity $\sqrt{X_1^2 + Y_1^2 + Z_1^2}$ of the elastic force developed is proportional to the relative displacement $\Delta\xi$, and since the axis has been assumed arbitrarily, it can be said, in general, that the total intensity, $\varepsilon\sqrt{X^2 + Y^2 + Z^2}$, developed, is directly proportional to the general relative displacement,

$$\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2} = \varepsilon.$$

From Eqs. (22) we conclude that the component intensity of the elastic force developed in the direction of any axis, due to any displacement, is equal to the sum of the three component intensities developed by three successive displacements along these axes, equal to the respective projections of the general displacement on these axes.

32. Of the nine coefficients of $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, given in Eqs. (18), six only are distinct. Representing these by

$$\left.\begin{aligned} A &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right], \\ B &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right], \\ C &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right], \\ D &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta y\,\Delta z}{r^2}\right], \\ E &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta y}{r^2}\right], \\ F &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta z}{r^2}\right], \end{aligned}\right\} \qquad (23)$$

we can write Eqs. (22),

$$\left.\begin{aligned} X\varepsilon &= A\,\Delta\xi + E\,\Delta\eta + F\,\Delta\zeta, \\ Y\varepsilon &= E\,\Delta\xi + B\,\Delta\eta + D\,\Delta\zeta, \\ Z\varepsilon &= F\,\Delta\xi + D\,\Delta\eta + C\,\Delta\zeta; \end{aligned}\right\} \qquad (24)$$

from which we conclude that the component elastic force developed along any axis, x, for example, by a displacement ε along any other axis y is equal to the component elastic force developed along the axis y by an equal displacement along the axis x.

33. From Eqs. (19–21) we see that when a displacement is made in any direction, the resulting elastic force is not, in general, in the same direction. To find whether we can refer the system to rectangular co-ordinate axes, so that when a displacement is made along such an axis, exceptional elastic forces will be developed, whose total resultant will be in the direction of the displacement, let α, β, γ, be the angles which the direction of the displacement makes with the axes; λ, μ, and ν, the angles which the resultant elastic force makes with the same axes; then we have

$$\left.\begin{aligned}
\cos\lambda &= \frac{X}{\sqrt{X^2 + Y^2 + Z^2}}, \\
\cos\mu &= \frac{Y}{\sqrt{X^2 + Y^2 + Z^2}}, \\
\cos\nu &= \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}, \\
\cos\alpha &= \frac{\Delta\xi}{\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}}, \\
\cos\beta &= \frac{\Delta\eta}{\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}}, \\
\cos\gamma &= \frac{\Delta\zeta}{\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}},
\end{aligned}\right\} \tag{25}$$

Since the resultant intensity of the elastic force is proportional to the displacement, we may let K represent the intensity of the elastic force corresponding to a displacement equal to unity. K varying with the direction of the displacement, we can then place $K\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}$ for its representative, $\varepsilon\sqrt{X^2+Y^2+Z^2}$, and the first of Eqs. (25) will become

$$\left.\begin{aligned}
\cos\lambda &= \frac{X\varepsilon}{K\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}}, \\
\cos\mu &= \frac{Y\varepsilon}{K\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}}, \\
\cos\nu &= \frac{Z\varepsilon}{K\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}},
\end{aligned}\right\} \tag{26}$$

Substituting the values of X, Y, Z, $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, derived from these equations in Eqs. (24), after omitting the common factor ε, we have

$$\left.\begin{aligned}
K\cos\lambda &= A\cos\alpha + E\cos\beta + F\cos\gamma, \\
K\cos\mu &= E\cos\alpha + B\cos\beta + D\cos\gamma, \\
K\cos\nu &= F\cos\alpha + D\cos\beta + C\cos\gamma.
\end{aligned}\right\} \tag{27}$$

Applying the conditions

$$\lambda = \alpha, \qquad \mu = \beta, \qquad \nu = \gamma,$$

we have the equations of condition,

$$\left.\begin{array}{l}(A - K)\cos\alpha + E\cos\beta + F\cos\gamma = 0,\\ E\cos\alpha + (B - K)\cos\beta + D\cos\gamma = 0,\\ F\cos\alpha + D\cos\beta + (C - K)\cos\gamma = 0;\end{array}\right\} \qquad (28)$$

together with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$ (29)

which make four equations containing the four unknown quantities α, β, γ, and K.

34. In order that Eqs. (28) may be true for the same set of values of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, we must have the determinant

$$\left\{\begin{array}{rrr} A - K, & E, & F,\\ E, & B - K, & D,\\ F, & D, & C - K,\end{array}\right\} = 0. \qquad (30)$$

Multiplying Eqs. (28) respectively by D, F, and E, we get

$$\left.\begin{array}{l}(AD - KD)\cos\alpha + DE\cos\beta + DF\cos\gamma = 0,\\ EF\cos\alpha + (BF - FK)\cos\beta + DF\cos\gamma = 0,\\ EF\cos\alpha + DE\cos\beta + (CE - EK)\cos\gamma = 0.\end{array}\right\} \qquad (31)$$

Placing

$$\left.\begin{array}{l}AD - EF = aD,\\ BF - DE = a'F,\\ CE - DF = a''E,\end{array}\right\} \qquad (32)$$

we have

$$\left.\begin{array}{l}(a - K)D\cos\alpha + EF\cos\alpha + DE\cos\beta + DF\cos\gamma = 0,\\ (a' - K)F\cos\beta + EF\cos\alpha + DE\cos\beta + DF\cos\gamma = 0,\\ (a'' - K)E\cos\gamma + EF\cos\alpha + DE\cos\beta + DF\cos\gamma = 0;\end{array}\right\} \qquad (33)$$

from which

$$\left.\begin{array}{l}(a - K)D\cos\alpha = (a' - K)F\cos\beta\\ \qquad = (a'' - K)E\cos\gamma = P;\end{array}\right\} \qquad (34)$$

whence,

$$\left.\begin{array}{l}\cos\alpha = \dfrac{P}{(a - K)D},\\[2ex] \cos\beta = \dfrac{P}{(a' - K)F},\\[2ex] \cos\gamma = \dfrac{P}{(a'' - K)E}.\end{array}\right\} \qquad (35)$$

Substituting these values in the first of Eqs. 33, we obtain

$$1 + \frac{EF}{(a-K)D} + \frac{DE}{(a'-K)F} + \frac{DF}{(a''-K)E} = 0. \quad (36)$$

Clearing of fractions, we have

$$\left. \begin{aligned} DEF(K-a)(K-a')(K-a'') - E^2F^2(K-a')(K-a'') \\ -D^2E^2(K-a'')(K-a) \\ -D^2F^2(K-a)(K-a') \end{aligned} \right\} = 0. \quad (37)$$

If DEF be positive, supposing

1°, that $a < a' < a''$, by substituting for K, in succession in Eq. (37), $-\infty$, a, a', a'', $+\infty$, we obtain

$$\begin{aligned} &-\infty, \\ &-E^2F^2(a'-a)(a''-a), \\ &+D^2E^2(a''-a')(a'-a), \\ &-D^2F^2(a''-a)(a''-a'), \\ &+\infty, \end{aligned}$$

which, since there are three variations in the signs, shows that Eq. (37) has three real roots, one lying between a and a', one between a' and a'', and the third between a'' and ∞. Similarly, if DEF be negative, the real roots will be found as above.

2°. If two of the quantities a, a', a'', are equal, as, for example, $a' = a''$, Eq. (37) reduces to

$$\left. \begin{aligned} (K-a')\{[EF(K-a')][D(K-a)-EF] \\ -D^2(F^2+E^2)(K-a)\} \end{aligned} \right\} = 0. \quad (38)$$

which gives a real root between a and a', a second equal to a', and a third greater than a'.

3°. If the three quantities a, a', a'', are equal, Eq. (37) reduces to

$$(K-a)^2[DEF(K-a) - E^2F^2 - D^2E^2 - D^2F^2] = 0, \quad (39)$$

giving two real roots, each equal to a, and one greater than a. Each of these real roots of K, being substituted in one of Eqs. (35), will enable us to find values for each of the cosines between $+1$ and -1, and hence a given direction for each value of K, or in all *three directions.*

35. We therefore conclude, *that the total elastic force developed by any displacement is not in general in the line of direction of the displacement, but oblique to it; that there are three directions at right angles to each other, and, in general, only three, along which, if the displacement be made, the resultant elastic force developed will be in the direction of the displacement.*

36. These three directions are called *principal axes.* They are not specific lines in a body, but simply mark directions along which the above property exists.

37. The angle which the direction of the displacement and the resultant elastic force make with each other is given by

$$\left.\begin{aligned}\cos U &= \frac{X\,\Delta\xi + Y\,\Delta\eta + Z\,\Delta\zeta}{\sqrt{X^2+Y^2+Z^2}\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}}\\ &= \frac{X\,\Delta\xi + Y\,\Delta\eta + Z\,\Delta\zeta}{K\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}};\end{aligned}\right\} \qquad (40)$$

and, if the displacement be equal to unity, we have

$$\left.\begin{aligned}K\cos U &= X\,\Delta\xi + Y\,\Delta\eta + Z\,\Delta\zeta\\ &= X\cos\alpha + Y\cos\beta + Z\cos\gamma.\end{aligned}\right\} \qquad (41)$$

38. ***Surfaces of Elasticity.*** If now distances which are proportional to the elastic forces developed by a constant displacement, equal to unity, for example, in each direction, be laid off in all directions from any point of the medium, the extremities of these lines will form a surface which may be called a *surface of elasticity.* But, as for each direction there are two things to consider, viz., the intensity of the elastic force and the angle which its direction makes with the displacement, we cannot, in general, construct a surface which would unite these two particulars.

39. It will be shown, hereafter, upon what grounds we can disregard, in optics, that component of the elastic force, $K \sin U$, which is perpendicular to the displacement, and consider, as alone effective, the component whose intensity is represented by $K \cos U$, parallel to the displacement.

40. Assuming then, for the present, that the effective elastic force caused by a displacement equal to unity is given by Eq. (41), and substituting the radius vector r for the first member, and the values of X, Y, Z, from Eqs. (24), and for $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, their values for a displacement unity, we get

$$\left.\begin{aligned} r = A \cos^2\alpha + 2E\cos\alpha\cos\beta + 2F\cos\alpha\cos\gamma \\ + B\cos^2\beta + 2D\cos\beta\cos\gamma + C\cos^2\gamma, \end{aligned}\right\} \quad (42)$$

the polar equation of a surface of elasticity of the medium.

Substituting for $\cos\alpha$, $\cos\beta$, $\cos\gamma$, their values $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$, and for r its equal $\sqrt{x^2+y^2+z^2}$, Eq. (42) becomes

$$\sqrt{x^2+y^2+z^2} = \frac{1}{x^2+y^2+z^2}[Ax^2+By^2+Cz^2+2Exy+2Fxz+2Dyz]. \quad (43)$$

41. Assuming that the radius vector is proportional to the square root of the elastic force, the equation takes the form

$$(x^2+y^2+z^2)^2 = Ax^2+By^2+Cz^2+2Exy+2Fxz+2Dyz, \quad (44)$$

which is the equation of *Fresnel's Surface of Elasticity.*

42. By assuming each radius vector proportional to the reciprocal of the square root of the elastic force, Eq. (42) becomes

$$1 = Ax^2 + By^2 + Cz^2 + 2Exy + 2Fxz + 2Dyz, \quad (45)$$

which is the equation of what has been designated as the *inverse ellipsoid of elasticity*, or the *first ellipsoid*, and is called the ellipsoid E.

43. ***Surfaces of Elasticity referred to Principal Axes.*** Principal axes are those along which, if the displacement be made, the resultant elastic forces developed will be wholly in the same direction. We have seen that, in any homogeneous medium, there are in general three, and only three, such directions. Making $\Delta\eta$, $\Delta\zeta$; $\Delta\xi$, $\Delta\zeta$; $\Delta\xi$, $\Delta\eta$, respectively equal to zero in Eqs. (24), and placing A, B, C, equal to a^2, b^2, c^2, respectively, we have

$$\left.\begin{aligned} X = a^2\Delta\xi, \qquad Y = b^2\Delta\eta, \qquad Z = c^2\Delta\zeta, \\ E = F = D = 0, \end{aligned}\right\} \quad (46)$$

and Eq. (44) reduces to

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2; \quad (47)$$

and Eq. (45) to $$a^2x^2 + b^2y^2 + c^2z^2 = 1. \quad (48)$$

Fresnel's surface of elasticity, Eq. (47), is of the fourth order, its equation being of the fourth degree. Figure (3) represents one-quarter of the principal section made by the plane *ac*, turned about the axis *b* through an angle of 90°. Taking the axes to be

$$a = 1.53, \qquad b = 1.32, \qquad c = 1.00,$$

we may, by Eq. (47), readily construct the principal sections. Thus, since

$$r^4 = a^2x^2 + b^2y^2 + c^2z^2, \qquad \therefore\ a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma = r^2;$$

we have for the intersection by the plane *ac*, $\beta = 90°$, and

$$r^2 = a^2 \cos^2 \alpha + c^2 \cos^2 \gamma = r'^2 + r''^2,$$

when $\qquad r' = a \cos \alpha, \quad$ and $\quad r'' = c \cos \gamma = c \sin \alpha.$

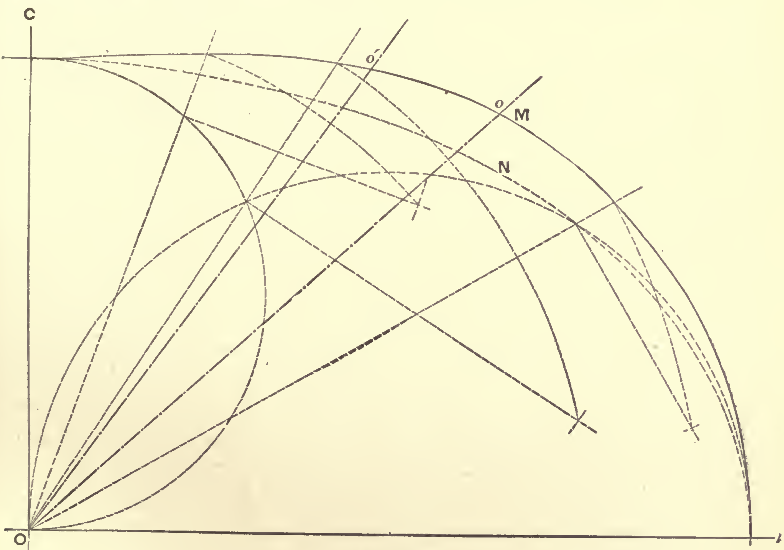


Figure 3.

Therefore r is equal to the hypothenuse of the right-angled triangle on r' and r''; hence, describe semicircles on a and c; draw any right line from O, and lay off on it a distance equal to the hypothenuse on the intercepts of the two circles, and this will be a point of the curve. Three such points are constructed in the figure. The curve CMA is the intersection of Fresnel's surface with *ac*; the curve CNA is that of the ellipsoid whose semi-axes coincide with and are equal to those of the surface of Fresnel; O*o* and O*o'* are the traces of the cyclic planes which contain the axis *b* of the surface of elasticity and of the ellipsoid respectively. The principal elasticities in crystals never differ so much as those assumed above, and therefore, in many cases, the departure of the surface from the ellipsoid is negligible.

WAVES.

44. The elastic forces of the medium, developed by the assumed arbitrary displacement of a molecule, will propagate the motion in all directions from the point of initial disturbance. As an ever-enlarging volume becomes involved in this disturbance, each molecule takes up a motion exactly similar to that of its predecessor, which it transmits in turn to the next molecule. This transfer is complete when a single pulse traverses the medium, and is both complete and continuous when these pulses are successively continuous.

In this latter case the exciting cause acts for a definite portion of time. Representing by a series of dots, *a....b*, the position of

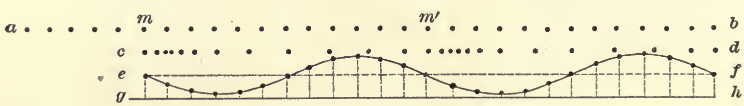


Figure 4.

a file of molecules in their condition of stable equilibrium and considering alone the simple case of rectilineal displacements, the arbitrary displacement of the molecule *m* will give rise to the successive displacements of the others, and *cd* and *ef* will represent the relative positions of these molecules at the end of a given subsequent time *t*, equal to the periodic time of vibration; the former, when the displacements are parallel to the direction of disturbance propagation, and the latter, when at right angles to this direction.

While, therefore, any molecule *m* is describing its orbit, the disturbance is being propagated in all directions, and, at the instant the orbit of *m* is completed, the disturbance will have reached another molecule m', on the same line of direction, which will then, for the first time, begin to move; and the molecules m and m' will, thereafter, always be at the same relative distances from their origins.

45. While this undulatory motion is being propagated, molecules will be found between m and m', with all degrees of displacement, both as to amount and direction of motion, consistent with the dimensions and shapes of their orbits. If the velocity of wave propagation be constant in all directions, the form assumed by the bounding surface containing the disturbed molecules will be spherical; but if the velocity vary, the form will depend upon the law of its variation.

46. This continuous transmission in any given direction of a *relative* state of the molecules, while the motion of each molecule is orbital, is characteristic of an *undulation.*

47. The term *phase* is used to express the condition of a molecule with respect to its *displacement* and the *direction* of its *motion.* Molecules are said to be in *similar* phases, when moving in parallel orbital elements and in the same direction; and in *opposite* phases, when moving in parallel orbital elements and in opposite directions. More generally, similar phases are those in which the anomalies of the molecule are the same, and opposite phases those in which the anomalies differ by 180°. (By *anomaly* is meant the angular distance from an assumed right line.)

48. A *wave* is the particular *form of aggregation* assumed by the molecules between the nearest two consecutive surfaces in which similar phases simultaneously exist throughout.

A *wave front* is that surface which contains molecules only in the same phase; it is generally understood to refer to the surface upon which the molecules are just beginning to move. The velocity of a wave front will always be that of the disturbance propagation.

A *wave length* is the interval, measured in the direction of wave propagation, between the nearest two consecutive surfaces upon which the molecules have similar phases.

The *amplitude* of the undulation is the maximum displacement of the molecule from its place of rest.

49. From a consideration of the nature of an undulation, we see at once that, if λ be the wave length, τ the periodic time, and V the velocity of wave propagation, we will have

$$V = \frac{\lambda}{\tau}, \tag{49}$$

and the values of V, λ, and τ are each, theoretically, independent of the amplitude.

50. To find an expression for the displacement of a molecule at any time during the transmission of an undulation, let x be the distance of the molecule from the origin of disturbance, t the time from the epoch, τ the periodic time of the molecule, λ the wave length, and V the velocity of wave propagation. Now, whatever be the displacement δ of the molecule x, at the time t, an equal displacement (neglecting the loss due to increased distance from the origin) will exist for another molecule at a distance $x + Vt'$, at the time $t + t'$. This condition gives, whatever be the value of t',

$$\delta = \phi(x, t) = \phi(x + Vt', t + t'). \tag{50}$$

$x + Vt'$ is the distance from the origin to the wave front at a time t subsequent to the instant at which it was at x. Hence the molecule x is behind the wave front a distance $Vt - x$, and the displacement, $\phi(x, t)$, may be replaced by $\phi(Vt - x)$; therefore we have

$$\delta = \phi(x, t) = \phi(Vt - x), \tag{51}$$

as the form of the function.

51. We have implicitly assumed the medium to be in a state of stable equilibrium during the passage of the undulation, and, therefore, the molecule will necessarily describe a closed orbit about its place of relative rest. This orbit may, from the circumstances of the case, be of the most varied character, and, after the energy due to the disturbance has been dissipated, the molecule will resume its original place of relative rest, until again displaced by some new disturbance. It is necessary, in this discussion, to consider those disturbances alone which are regular and periodic, and to consider the orbit after it has become determinate. We therefore limit the discussion to that of the regular periodic disturbance, and the orbit to that of the ellipse or any of its particular cases, such as the ellipse, the circle, or the right line.

52. ***Simple Harmonic Motion.*** If a point a (Fig. 5) move uniformly in a circular orbit, the distance of its projection from the centre, upon the vertical diameter, can always be found from the equation

$$y = a \sin\left(\frac{2\pi t}{\tau} + \alpha\right), \qquad (52)$$

in which y is the required displacement at the time t, a is the amplitude or maximum displacement, τ the periodic time, and α the angle included between the horizontal diameter and that passing through the origin of motion.

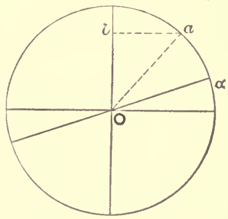


Figure 5.

The angle $\frac{2\pi t}{\tau} + \alpha$ is called the phase of the vibration, and may be made of any value by changing the arbitrary arc α, the time t, or both together. The same value will apply to motion along any diameter. Such motions are called simple harmonic motions. It may easily be shown that any two simple harmonic motions, in one line and of the same period, may be compounded into a single simple harmonic motion of the same period, but whose amplitude is equal to the diagonal of a parallelogram constructed on the amplitudes of the components inclined to each other by an angle equal to their difference of phase.

53. ***The Harmonic Curve.*** If the motion of a point be compounded of a rectilineal harmonic vibration, and of uniform motion in a straight line perpendicular to the vibration, the point will describe a plane curve, which is called the *harmonic curve.*

Let the vibration be along the axis of y, and uniform motion along the axis x; we will then have

$$y = a \sin\left(\frac{2\pi t}{\tau} + \alpha\right) \qquad (53)$$

for the ordinates, and

$$x = Vt \qquad (54)$$

for the abscissas, due to the uniform motion. Combining these equations, eliminating t, and replacing $V\tau$ by its equal λ, Eq. (49), page 40, we have, for the equation of the harmonic curve,

$$y = a \sin\left(\frac{2\pi x}{\lambda} + \alpha\right); \qquad (55)$$

in which λ is the wave length. Substituting for x, $x \pm i\lambda$, the value of y remains the same for all integral values of i. The curve, therefore, consists of an infinite number of similar parts, which are symmetrical with respect to the axis of x.

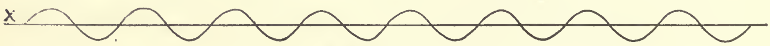


Figure 6.

54. To construct the curve by points, divide the circumference into any number, as twelve, equal parts; lay off on the axis of abscissas twelve equal distances, corresponding to the positions of the point in uniform motion, erect ordinates at these points and make them equal to the corresponding displacements at the given times, and we have the curve as follows:

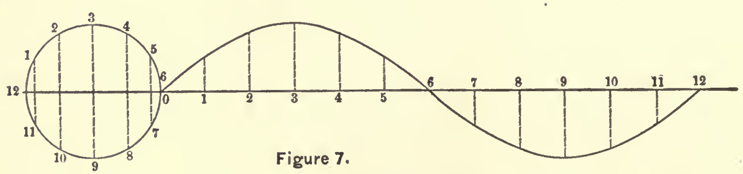


Figure 7.

55. The varying velocities of a point of a simple pendulum in motion can be represented by the ordinates of the harmonic curve; and because of this analogy all vibrations represented by these curves are called simple or pendular vibrations. The vibration is taken to be the complete oscillation, from the time at which the moving point was in one position until it returns to the same position again. By this definition, the duration of the vibration of a second's pendulum would be *two* seconds, and not *one* second.

56. ***Composition of Harmonic Curves.*** Let

$$y' = a \sin\left(\frac{2\pi x}{\lambda} + \alpha\right), \tag{56}$$

$$y'' = b \sin\left(\frac{2\pi x}{\lambda} + \beta\right), \tag{57}$$

be the equations of any two harmonic curves, having the same wave length, but different amplitudes. The resultant value of y will be

$$y = c \sin\left(\frac{2\pi x}{\lambda} + \gamma\right), \tag{58}$$

which is the equation of another harmonic curve, of equal wave length, but of different amplitude from either of the components.

The values of c and γ are given by

$$c \cos \gamma = a \cos \alpha + b \cos \beta, \tag{59}$$

$$c \sin \gamma = a \sin \alpha + b \sin \beta, \tag{60}$$

$$c = \sqrt{a^2 + b^2 + 2ab \cos(\alpha - \beta)}. \tag{61}$$

From the last equation we see that c may have any value between the sum and difference of a and b, depending upon the value of the difference of phase, $\alpha - \beta$, of the components.

By a similar process, it can be shown that any number of component harmonic curves, of the same wave length, may be compounded into a single resultant harmonic curve having an equal wave length, but whose amplitude and phase differ in general from those of any of its components.

57. If the component curves have different wave lengths, they cannot be compounded into a single harmonic curve; but when their wave lengths are commensurable, they can be compounded into a periodic curve, whose period is the least common multiple of their several periods. Thus, in the first case, where the wave lengths are unequal and incommensurable for the resultant ordinate,

$$y = a \sin\left(\frac{2\pi x}{\lambda'} + \alpha\right) + b \sin\left(\frac{2\pi x}{\lambda''} + \beta\right) + c \sin\left(\frac{2\pi x}{\lambda'''} + \gamma\right) + \ldots, \tag{62}$$

in which the period is infinite, or the curve is non-periodic.

In the second case, let

$$\lambda' = \frac{\lambda}{m}, \qquad \lambda'' = \frac{\lambda}{n}, \qquad \lambda''' = \frac{\lambda}{r}, \ \ldots\ldots\ldots, \tag{63}$$

m, n, r, being integers; then the above equation becomes

$$y = a \sin\left(\frac{2\pi m x}{\lambda} + \alpha\right) + b \sin\left(\frac{2\pi n x}{\lambda} + \beta\right) + c \sin\left(\frac{2\pi r x}{\lambda} + \gamma\right) + \ldots, \tag{64}$$

which, although not admitting of reduction to a simpler form, gives constantly recurring values of y when for x we substitute $x + \lambda$. The wave length of the resultant curve is therefore λ, and the curve is periodic.

58. The *forms* of the component curves depend only upon the wave lengths and amplitudes; but their positions on the axis depend on the values of the phase α, β, γ, etc. By assigning arbitrary values to these, we may shift any curve along the axis any desired part of its wave length. Any such shifting for any one or more of the component curves will necessarily alter the form of the resultant curve, but will not change its wave length.

59. If the wave length of the resultant curve be assumed, the wave lengths of its components may be all possible aliquot parts of λ, and the number of the possible components is therefore unlimited. Therefore every possible curve of wave length λ, which could be so constructed from such component curves, would be found among those produced by placing, along the same axis, an unlimited number of harmonic curves, as components, with wave lengths λ, $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda$, etc., ...

By varying the amplitudes of the components and shifting them arbitrarily along the axis, an infinite number of resultants can be produced, all having the same wave length λ. Fourier's theorem demonstrates that every possible variety of periodic curve, of given wave length λ, can be so produced, provided that the ordinate is always finite and that the moving point is assumed to move always in the same direction.

60. A periodic series is one whose terms contain sines or cosines of the variable, or of its multiples; thus,

$$A_1 \cos x + A_2 \cos 2x + A_3 \cos 3x + \ldots A_n \cos nx + \ldots$$

is a periodic series. This series goes through a succession of values as the arc increases from 0 to 2π; for, every term has the same value at the end and at the beginning of that period, and this continuously, so that whatever n may be, the period of the function is 2π.

61. Fourier's Theorem has for its object the determination of the unknown constants, $A_0, A_1, A_2, \ldots B_1, B_2, B_3, \ldots$, and the determination of the conditions by which any given function, $y = f(x)$, can be expressed in the form of

$$\left.\begin{aligned} y = f(x) = A_0 + A_1 \cos x + A_2 \cos 2x + \ldots \\ + B_1 \sin x + B_2 \sin 2x + \text{etc.} \ldots \end{aligned}\right\} \quad (65)$$

The non-periodic term A_0 is introduced to make the theorem conform to the most general case. If the function is capable of expression in periodic terms only, then $A_0 = 0$; this fact can only be determined by considering each special case.

The equation which expresses the mathematical statement of Fourier's Theorem is

$$y = y_0 + \Sigma_{i=1}^{i=\infty} C_i \sin\left(\frac{2i\pi t}{\tau} + \alpha_i\right), \quad (66)$$

in which y_0 is the mean value of y, and each of the variable terms represents, by itself, a harmonic vibration of which the period is an aliquot part of the whole period τ.

62. *Wave Function.* Resuming Eq. (51),

$$\delta = \phi(x, t) = \varphi(Vt - x),$$

we see that, since the displacement δ passes through all of its values while the undulation advances a distance equal to its wave length λ, it has the properties of simple harmonic motion, and, therefore, may be written

$$\delta = a \sin \frac{2\pi}{\lambda}(Vt - x). \quad (67)$$

This is called the wave function. By making t vary continuously through all values from $t = \frac{x}{V}$ to $t = \frac{x + \lambda}{V}$, δ will increase from zero to $+a$, decrease then to $-a$, and finally return to zero, during the time $\frac{\lambda}{V}$, which is evidently the interval of time required for the undulation to pass over the wave length λ. Again, supposing t to remain constant and x to vary through all values from $Vt - \lambda$ to Vt, we obtain again all possible values of the displacement, which values will evidently belong, at the same instant, to all molecules in the wave length. The following diagram illustrates the two cases:

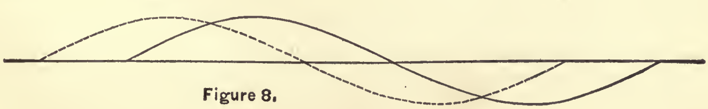


Figure 8.

By the addition of an arbitrary arc we can cause the displacement to take any one of its values, at any time t, and thus change our origin at pleasure.

63. The corresponding expression for the velocity of the molecule in its rectilinear orbit, sometimes called the *velocity* of the *wave element*, in contradistinction to the *velocity* of *wave propagation*, is given by

$$u = a \cos \frac{2\pi}{\lambda}(Vt - x). \qquad (68)$$

64. The principle of the coexistence and superposition of small motions is shown in Mechanics to be applicable to planetary perturbations. It is, for similar reasons, applicable to the determination of the resultant displacement of a single molecule, arising from the concurrent effect of many disturbing causes acting separately. The acceptance of this principle is equivalent to assuming that the several displacements are so small that their products and powers higher than the first are negligible with respect to the displacements themselves; and it embodies the primary supposition that the intensity of elastic forces developed varies directly with the degree of displacement.

65. ***Wave Interference.*** If we apply this principle to determine the displacement of a molecule by two disturbing causes, giving rise to two undulations of the same wave length, we will have for the first,

$$\delta' = a' \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A'\right]; \qquad (69)$$

for the second,

$$\delta'' = a'' \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A''\right]. \qquad (70)$$

The total displacement will be

$$\left.\begin{aligned}\delta' + \delta'' = \delta = (a' \sin A' + a'' \sin A'') \cos\left[\frac{2\pi}{\lambda}(Vt - x)\right] \\ + (a' \cos A' + a'' \cos A'') \sin\left[\frac{2\pi}{\lambda}(Vt - x)\right],\end{aligned}\right\} \qquad (71)$$

which may be put under the form

$$\delta = a \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A\right], \qquad (72)$$

by placing
$$\left.\begin{aligned} a \cos A &= a' \cos A' + a'' \cos A'', \\ a \sin A &= a' \sin A' + a'' \sin A''. \end{aligned}\right\} \qquad (73)$$

Whence,
$$a^2 = a'^2 + a''^2 + 2a'a'' \cos (A' - A''), \qquad (74)$$

$$\tan A = \frac{a' \sin A' + a'' \sin A''}{a' \cos A' + a'' \cos A''}. \qquad (75)$$

By Eq. (72) we see that the resultant undulation is of the same wave length as the components; that the maximum displacement of the resultant undulation is not, in general, equal to that of either of the components, and that it does not occur at the same time nor place with either of them.

66. Taking the square root of Eq. (74), we have

$$a = \sqrt{a'^2 + a''^2 + 2a'a'' \cos (A' - A'')}\,; \qquad (76)$$

from which it is seen that, when $A' - A'' = 0$, $a = a' + a''$; and, when $A' - A'' = 180°$, $a = a' - a''$. Hence, in Eq. (75), $A = A' = A''$ in the first case, and $A = A' = 180° + A''$ in the second. The maximum displacement, then, of the resultant undulation may vary between the sum and difference of the maximum displacements of the two component undulations, depending upon the difference of phase.

If, in the two component undulations, $a' = a''$, a will be equal to $2a'$ when $A' = A''$, and vary from this value to zero as the difference of phase $A' - A''$ passes from zero to 180°.

Substituting, in the expression for the displacement, $A' \pm 180°$ for A', we will have

$$a' \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A' + \pi\right] = a' \sin\left[\frac{2\pi}{\lambda}\left(Vt - x \pm \frac{\lambda}{2}\right) + A'\right], \quad (77)$$

which is exactly the same as

$$a' \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A'\right],$$

when for x we put $x \mp \frac{\lambda}{2}$.

Therefore, if we suppose that two undulations of the same wave length, starting in the same phase, meet after travelling over routes which differ by one-half the wave-length, there will be no displacement of the molecule at the place of meeting, and complete interference will result.

The diagrams of Figure 9 illustrate the composition of two undulations of equal wave length, having the same phase in the first case, and opposite phases in the second and third cases. In AB, the amplitude of the resultant undulation a is equal to the sum of the amplitudes of the component undulations, a' and a''; in A'B' and A''B'', equal to the difference of the amplitudes. In A''B'', the displacement of the molecules is zero, and the two components mutually destroy each other's action.

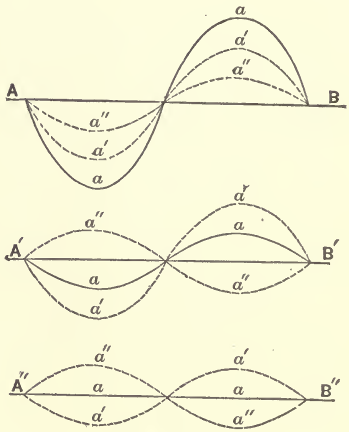


Figure 9.

67. *Interference of any Number of Undulations.*

1° Case. When the component undulations have the same wave length.

Let

$$\left.\begin{aligned} \delta' &= \alpha' \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A'\right], \\ \delta'' &= \alpha'' \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A''\right], \\ \delta''' &= \alpha''' \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A'''\right], \\ \text{etc.,} & \qquad \text{etc.,} \ldots, \end{aligned}\right\} \quad (78)$$

be the values of the several component displacements. By addition we have

$$\left.\begin{aligned} &\delta' + \delta'' + \delta''' + \text{etc.} \\ &= (\alpha' \sin A' + \alpha'' \sin A'' + \alpha''' \sin A''' + \text{etc.}) \cos\left[\frac{2\pi}{\lambda}(Vt - x)\right] \\ &+ (\alpha' \cos A' + \alpha'' \cos A'' + \alpha''' \cos A''' + \text{etc.}) \sin\left[\frac{2\pi}{\lambda}(Vt - x)\right]. \end{aligned}\right\} (79)$$

The second member may be placed under the form of

$$\left.\begin{aligned} a \sin A \cos \frac{2\pi}{\lambda}(Vt - x) + a \cos A \sin \frac{2\pi}{\lambda}(Vt - x) \\ = a \sin \left[\frac{2\pi}{\lambda}(Vt - x) + A\right] = \delta. \end{aligned}\right\} \quad (80)$$

From which we conclude that the resultant undulation will have the same wave length as that of the components, but that in general the maximum displacement and the phase at the time t will be different from those of its components.

68. 2° CASE. Component undulations of different wave lengths.

If the wave lengths are different, the displacements are of the form

$$a' \sin \left[\frac{2\pi}{\lambda}(Vt - x) + A'\right],$$

$$a'' \sin \left[\frac{2\pi}{\lambda'}(Vt - x) + A''\right],$$

which cannot be combined into a single circular function of the same form. If in addition the wave velocities also differ, they may be combined if $\frac{V}{\lambda} = \frac{V'}{\lambda'}$. Hence, undulations of different wave lengths cannot destroy each other, and the combined effect of several undulations upon a single molecule will be equal to the algebraic sum of their separate effects. If this sum should reduce to zero for a given molecule, it will differ from zero for the molecules immediately preceding and following it.

69. ***The Principle of Huyghens.*** Since the displacement of any molecule is the cause of the subsequent displacement of other molecules, we may regard the displacement of the molecules upon any wave front as the cause of the subsequent displacement of the molecules upon any other front which the wave afterwards reaches. We may therefore consider each molecule of the wave front in any of its anterior positions as being the origin, and its displacement as the cause of secondary waves, each of which proceed with the same velocity. The aggregate effect of all these

secondary waves upon any other molecule beyond, or its resultant displacement, will evidently be the same as that due to the primary wave itself. This principle is known as that of Huyghens, and, together with the principle of interference, is exceedingly fruitful in explaining many of the phenomena of wave motion in sound and light.

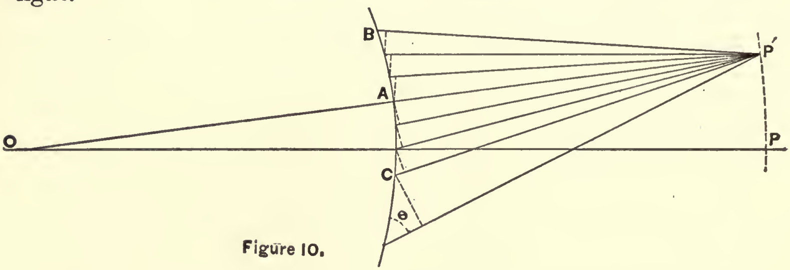


Figure 10.

Let O be the origin of disturbance, and BAC the great wave in any of its anterior positions before reaching a molecule P′; let AP′ $= l'$, AB $=$ AC $= b$; let dz be any indefinitely small part of the wave front, and θ the angle made by the wave front with any right line l drawn from P′ to any point of the wave front, at a distance z from A; then

$$l' = \text{AP}' = \sqrt{(l^2 - 2lz\cos\theta + z^2)} \tag{81}$$

$$= l - z\cos\theta + \frac{z^2}{2l}\sin^2\theta + \text{etc.} \tag{82}$$

$$= l - z\cos\theta, \tag{83}$$

for all points of BAC near to A, and for which $\frac{z^2}{2l}$ is insignificant.

The displacement at P′, due to the secondary waves originating in dz, will therefore be

$$\delta' = \frac{\alpha\, dz}{\text{AP}'}\sin\frac{2\pi}{\lambda}(Vt - l + z\cos\theta). \tag{84}$$

Replacing AP′ by l, and integrating, we have for the resultant displacement of P′ due to the great wave,

$$\left.\begin{aligned}\delta = \Sigma\delta' &= \frac{\alpha}{l}\int dz\sin\frac{2\pi}{\lambda}(Vt - l + z\cos\theta)\\ &= -\frac{\alpha\lambda}{2\pi l\cos\theta}\cos\frac{2\pi}{\lambda}(Vt - l + z\cos\theta);\end{aligned}\right\} \tag{85}$$

and between the limits corresponding to $+b$ and $-b$,

$$\delta = \frac{a\lambda}{\pi l \cos\theta} \sin\frac{2\pi b\cos\theta}{\lambda} \sin 2\pi\frac{Vt - l}{\lambda}. \tag{86}$$

The maximum displacement is therefore

$$\delta = \frac{a\lambda}{\pi l \cos\theta} \sin\frac{2\pi b\cos\theta}{\lambda}. \tag{87}$$

70. 1°. The above value of the displacement will vary with b, θ, λ, and l. When, as in sound, λ is very great as compared with b, $\frac{2\pi b\cos\theta}{\lambda}$ will be so small that the arc may be substituted for the sine without material error, and

$$\delta = \frac{2ab}{l}, \tag{88}$$

which is independent of θ.

2°. When λ is much smaller than b, as in the case of light, we have, when $\cos\theta$ is very small, and θ therefore differs but little from 90°, again

$$\delta = \frac{2ab}{l}. \tag{89}$$

At other points, where θ is not great and $\cos\theta$ not small, the resultant displacement becomes equal to zero when

$$\frac{2\pi b\cos\theta}{\lambda} = \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi, \quad \text{etc.};$$

that is, when $\cos\theta = \pm\frac{\lambda}{2b}, \quad \pm\frac{2\lambda}{2b}, \quad \pm\frac{3\lambda}{2b}, \quad \text{etc.}$

The greatest resultant displacement, other than that indicated above, will be found by making in Eq. (87),

$$\sin\frac{2\pi b\cos\theta}{\lambda} = \pm 1, \tag{90}$$

and it will be equal to $\frac{a\lambda}{\pi l \cos\theta}$;

and, since the intensity of the sensation is directly proportional to the square of the maximum displacements, we will have the relation of the intensities,

$$\frac{a^2\lambda^2}{\pi^2 l^2 \cos^2\theta} : \frac{4a^2b^2}{l^2} = \lambda^2 : 4\pi^2 b^2 \cos^2\theta = \frac{\lambda^2}{4\pi^2 b^2 \cos^2\theta} : 1. \quad (91)$$

71. In acoustics it will be shown that the wave lengths corresponding to audible sounds will vary from $\frac{1140'}{20} = 57'$ to $\frac{1140'}{40000}$ $= \frac{1}{3}$ of an inch, and therefore there will be no point exterior to an aperture where the displacement will not occur, and hence the corresponding sound be heard. In light, the wave lengths vary between .000026 and .000017 of an inch, and there will be, according to the 2° case, alternations of light and darkness surrounding the central line drawn from the place of original disturbance to the centre of the aperture. These zones are called Huyghens' zones, and will be again referred to in the subject of diffraction.

72. ***Diffusion and Decay of Kinetic Energy.*** The displacement of any molecule due to wave motion of a given wave length is independent of the periodic time, and, since the orbits of the molecules are described in equal times when they arise from a given periodic motion, they will be directly proportional to the displacements or any other homologous lines. The velocities, then, of the moving molecules being represented by v, their kinetic energies will be represented by $\frac{mv^2}{2}$. Then, because these energies are transmitted without appreciable loss from the molecules of one surface to those of another, we will have the energies of the molecules of the two homologous surfaces,

$$4\pi r^2 \cdot \frac{mv^2}{2} = 4\pi r'^2 \cdot \frac{mv'^2}{2}, \quad \text{or} \quad \frac{mv^2}{2} \cdot r^2 = \frac{mv'^2}{2} \cdot r'^2; \quad (92)$$

that is, $\frac{mv^2}{2} : \frac{mv'^2}{2} :: r'^2 : r^2$, or varying according to the law of the inverse square of the distance. Similarly, we will have

$$\delta'' r'' = \delta' r', \quad (93)$$

or the maximum displacements inversely proportional to the distances to which the disturbance has been propagated.

73. *Reflection and Refraction.* It is difficult to conceive, satisfactorily, in what manner the molecules belonging to two media of different elasticity and density are arranged with respect to each other in or near the bounding surface which separates them. When they occupy positions of relative rest, the elastic forces must be mutually counterbalanced and must be equal to those affecting the molecules within the media. We may assume the two media to have different densities and elasticities, and the relative positions of the molecules near the separating surface to be determined by the action of the equilibrating molecular forces. But when a disturbance arising in one of the media reaches the surface, the molecules of the second medium must, in general, have motions and displacements different from those of the first. If we consider alone the difference in density of the molecules of the media, we perceive that the energy in the incident wave will not be wholly given up by the molecules to their neighbors in the new medium. In either case, whether the molecules have greater or less density, a return wave will originate in the incident medium, analogous to the reflected motion in the impact of elastic balls. Again, if the elasticity of the media be different, the elastic forces for equal displacements will be different, and thus cause a return wave in the incident medium. We may therefore assume, for the present, that, owing to the different elasticities or densities, or both, there will be, in general, a separation of the incident wave whenever it meets a surface separating two media of different density and elasticity. The fact of such a separation is experimentally demonstrated in the phenomena of sound and light. The velocity of wave propagation will be shown to be a function of the elasticity and density of the medium, and therefore the waves, in general, will proceed in the two media with different velocities.

74. The *plane of incidence* is that plane which is normal to the deviating surface and to the wave front.

The *plane of reflection* is normal to the deviating surface and to the reflected wave front; it coincides with the plane of incidence.

The *plane of refraction* is normal to the refracted wave front and to the deviating surface.

75. *Diverging, Converging, and Plane Waves.* When the energy of molecular disturbance is distributed among the molecules, upon an increasing wave front, the wave is said to be *diverging;* when among those of a decreasing wave front, a *converging* wave; and when among those of an unchanged wave front, a *plane* wave. An indefinitely small portion of the front of any diverging wave, taken at a correspondingly great distance from the origin, may, without sensible error, be considered as coinciding throughout with the tangent plane to the wave front, and considered as a plane wave. The molecules of a plane wave at any assumed position are animated by equal parallel displacements, and undergo all their phases while the plane wave advances a distance equal to the wave length, measured in a direction perpendicular to the plane.

76. Differentiating Eq. (67), we have

$$\frac{d^2\delta}{dt^2} = -\frac{4\pi^2 V^2}{\lambda^2}\delta. \tag{94}$$

Multiplying both members by m, the mass of the molecule, and replacing $m\frac{d^2\delta}{dt^2}$ by its equal $U\delta$, the intensity of the elastic force developed by the displacement δ, we have

$$U\delta = m\frac{d^2\delta}{dt^2} = -\frac{4\pi^2 m}{\lambda^2}V^2\delta; \tag{95}$$

whence,

$$U = -\frac{4\pi^2 m}{\lambda^2}V^2. \tag{96}$$

Hence, *when a plane wave is propagated without alteration in a homogeneous medium, its velocity of propagation is directly proportional to the square root of the elastic force developed by the displacement of its molecules.*

77. *Reflection and Refraction of Plane Waves.* Let the incident plane wave AC (Fig. 11) meet the deviating surface at all points, in succession, from A to B. Let V and λ be the velocity of wave propagation and the wave length in the medium of incidence, and V' and λ' those in the medium of intromittance. Let AB $= ds$, and CB $= V dt$. While the disturbance in the incident wave is moving from C to B, the disturbance from A as a centre will proceed in all directions in the medium of incidence, and be found, at the instant considered, upon the hemisphere whose radius is AD = CB = $V\,dt$, and in the medium of intromittance on the hemisphere whose radius is AD′ = $V'\,dt$.

Each point in the line AB will, in like manner, become in succession a new centre of disturbance, sending secondary waves into the media of incidence and of intromittance, whose radii will, at the instant the incident wave reaches B, be equal to V and V' multiplied by the interval of time elapsing between the instant of arrival of the wave front at the centre considered and that of its arrival at B. The surface through B, which is tangent to all the reflected pulses, may be taken as the front of the reflected wave, for it will contain more energy than any other surface of equal area in the incident medium. Similarly, the surface through B tangent to all the refracted pulses will contain more energy than any other of equal area in the medium of intromittance, and may be taken as the front of the refracted wave at this instant. These surfaces are readily seen to be planes; hence, denoting the angle CAB = ABD by ϕ, and ABD′ by ϕ', we will have

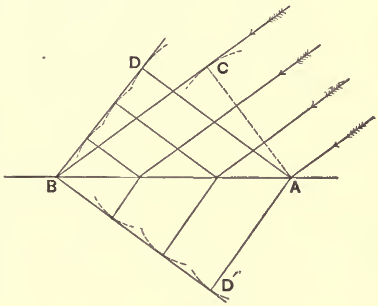


Figure 11.

$$ds \sin \phi = V\,dt, \qquad ds \sin \phi' = V'\,dt; \tag{97}$$

from which we obtain

$$\sin \phi = \frac{V}{V'} \sin \phi' = \mu \sin \phi', \tag{98}$$

which is known as Snell's law of the sines; μ is called the index of refraction.

78. The angles ϕ and ϕ' made by the wave fronts with the deviating surface are, respectively, equal to the angles made by the normals to the incident and refracted waves with the normal to the deviating surface, and

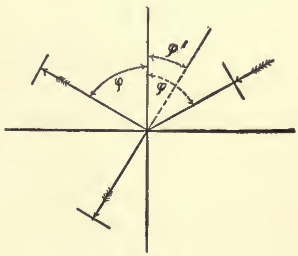


Figure 12.

are called *angles of incidence* and *refraction.* The angles of incidence and refraction are measured from the normal to the deviating surface on the side of the medium of incidence to the normal of the incident wave, and to that of the refracted wave produced back into the medium of incidence.

The angle of reflection is measured from the normal to the deviating surface to the normal to the reflected wave front, and is therefore negative. In the reflected wave, since the velocity of wave propagation is unchanged, μ is equal to unity, and Eq. (98) becomes

$$\sin \phi = -\sin \phi'. \tag{99}$$

79. These principles may, in ordinary cases, then be summarized as follows:

1°. The planes of incidence, reflection, and refraction are coincident.

2°. The sine of the angle of incidence is equal to the index of refraction or of reflection multiplied by the sine of the angle of refraction or of reflection.

The modifications which take place in polarized light will be referred to hereafter in physical optics.

80. We see from Art. 77 that the reflected and refracted waves are plane when an incident plane wave meets a plane deviating surface. It is evident also, from the construction, that the reflected rays are all normal to a plane NN′ symmetrical with MM′ with reference to OX; and that the incident and reflected rays are directed from their corresponding planes towards the deviating surface. The refracted rays are normal to a plane RR′ on the same side of the deviating surface as the incident wave, and are also directed towards that surface.

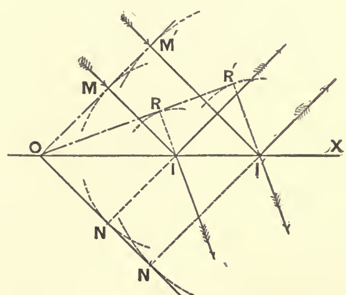


Figure 13.

81. ***General Construction of the Reflected and Refracted Waves.*** Let the deviating surface AB (Fig. 14) be any whatever, and the rays proceed from any origin O; take, in the medium of incidence, any spherical surface SS′, with centre at O, as the incident diverging wave; then, from all points I, I′, I″, etc., of AB, describe spheres, whose radii are equal to the intercepts of the rays between SS′ and AB. If, now, tangent planes be drawn to the deviating surface at I, I′, I″, etc., and to the surface SS′ at the corresponding points *s*, *s*′, *s*″, etc., each pair of tangent planes will determine, by their intersection, a right line, through which if a plane be passed tangent to the corresponding sphere on the other side of the deviating surface, it will be symmetrical with the infinitesimal surface of SS′ at *s* with respect to that of AB at the point I; and similarly for the other points. By continuity, these points of tangency may be considered as forming the envelope of the reflected wave. The direction of the reflected rays is found by joining these points with I, I′, I″, etc., and extending the lines toward and beyond the deviating surface.

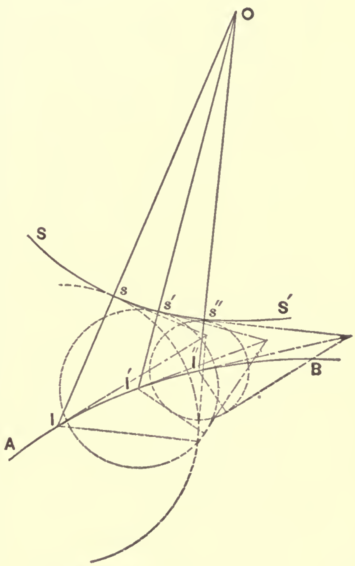


Figure 14.

82. By the proper modification of the radii due to the value of μ, the index of refraction, the envelope of the refracted wave and the direction of the refracted rays may be constructed.

83. Considering the reflected wave as a new incident wave, the new reflected wave, by another deviating surface, can be constructed by an application of the above principles; and since reflection may be considered as refraction whose index is -1, the principle may be generally stated, that any number of reflections and refractions may be replaced by a single refraction at a supposable deviating surface with a properly modified index of refraction.

84. Let DEF (Fig. 15) be any incident wave whose rays are not necessarily parallel; MNP any deviating surface. At some subsequent time t the incident wave will occupy some position such as ABG, FG being equal to $EB = DA = Vt$. By the principle established above, *abg* will be the enveloping surface of the reflected wave corresponding to ABG, and $a_1b_1g_1$ that of the refracted wave, and both will be concurrent, that is, the phases of the molecular motions on them will be similar; gPG′, bNB′, aMA′ will be the reflected, and g_1PG$_1$, b_1NB$_1$, a_1MA$_1$ the refracted rays.

85. Prolong the consecutive rays of either the reflected or refracted waves, say the reflected wave *abg*, until they meet two and two; they will be tangent to the surface $\alpha\beta\gamma$, which is the evolute of *abg*. Since the reflected rays are all normal to *abg*, this evolute will correspond to any other position of the reflected wave, also. The surface of which $\alpha\beta\gamma$ is a generatrix is in optics called the *caustic surface.* It is evident that the points of this caustic are not concurrent, because their distances, being equal to the radii of curvature of *abg* from the reflected wave, are themselves unequal; and points, in order to be concurrent, must be at equal distances from the wave surface. Whether the caustic be real or virtual, the displacements of its molecules being either due to that of two rays, or apparently so, the energy of the molecules, and hence the resulting sensation, will be greater than that due to but one ray.

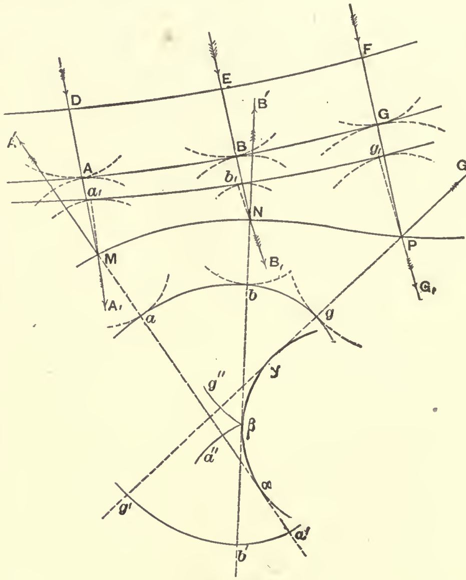


Figure 15.

86. When the evolute $\alpha\beta\gamma$ is known, the various possible positions of the reflected wave can readily be determined. In the ordinary cases considered in optics, the surfaces *abg* are those of revolution; the caustic is then also a surface of revolution. Suppose *abg* to be one of the generatrices of the reflected wave, considered as a surface of revolution, and $\alpha\beta\gamma$ to be its evolute; then, by the property of the evolute, if the tangent $a\alpha a'$ be caused to roll on $\alpha\beta\gamma$, each point of this tangent will describe one of the sections of the reflected wave. Thus, $a'b'g'$, $a''\beta g''$, and abg are such sections; the second of these being of two nappes, tangent to each other and normal to the evolute at the point β.

87. The principle that the rays, after the wave has been subjected to any number of reflections and refractions, are all normal to a theoretically determinable surface, and consequently to a series of surfaces, of which any two intercept the same length on all the rays, is principally applicable to the determination of caustic surfaces, and to the formation of optical images, and will therefore be further discussed in that branch of the subject.

88. ***Utility of Considering the Propagation of the Disturbance by Plane Waves.*** In a homogeneous medium, the arbitrary displacement of a molecule gives rise to elastic forces whose intensities depend on the degree and the direction of the displacements, and whose directions are not, in general, those of the displacements. In Art. (35) we have seen that the displacements must be made only in exceptional directions, in order that the elastic forces varying directly with the degree of the displacement should be wholly in those directions. Should the orbit of the displaced molecule be curvilinear, it is evident that, at each point of its path, the elastic forces developed would vary both in direction and intensity, and thus the general problem becomes one of extreme intricacy.

89. If, however, it be possible to limit the discussion to that of molecules in the same plane, all actuated by equal and parallel displacements, the variation as to direction of the elastic forces may, perhaps, be eliminated. It has been shown, Art. 76, that when a plane wave is propagated without alteration in a homogeneous medium, the velocity of propagation is directly proportional to the square root of the elastic force developed by the displacement. Hence the importance of deducing from the general equations (18) the corresponding equations applicable to the vibratory motions propagated by plane waves.

90. At the time t let r be the distance of the plane wave, in a homogeneous medium, from the origin of co-ordinates; ε the displacement of the molecules whose co-ordinates are x, y, z; ξ, η, ζ, the projections of ε on the rectangular co-ordinate axes; and α, β, γ, the angles made by the displacement with the axes, respectively.

We then have
$$\varepsilon = \delta \sin \frac{2\pi}{\lambda}(Vt - r); \tag{100}$$

$$\left.\begin{aligned} \xi &= \delta \cos \alpha \sin \frac{2\pi}{\lambda}(Vt - r); \\ \eta &= \delta \cos \beta \sin \frac{2\pi}{\lambda}(Vt - r); \\ \zeta &= \delta \cos \gamma \sin \frac{2\pi}{\lambda}(Vt - r). \end{aligned}\right\} \tag{101}$$

Let $r + \Delta r$ be the distance of the plane at a subsequent instant from the origin, and l, m, n, the angles made by the normal to the plane with the axes, then

$$r = x \cos l + y \cos m + z \cos n, \tag{102}$$

$$\Delta r = \Delta x \cos l + \Delta y \cos m + \Delta z \cos n. \tag{103}$$

From Eq. (101) we have

$$\left.\begin{aligned} \xi + \Delta\xi &= \delta \cos \alpha \sin \frac{2\pi}{\lambda}(Vt - r - \Delta r) \\ &= \delta \cos \alpha \Big[\sin \frac{2\pi}{\lambda}(Vt - r) \cos \frac{2\pi}{\lambda}\Delta r \\ &\qquad - \cos \frac{2\pi}{\lambda}(Vt - r) \sin \frac{2\pi}{\lambda}\Delta r \Big]; \end{aligned}\right\} \tag{104}$$

from which, and similarly for the axes y and z, we have

$$\left.\begin{aligned} \Delta\xi &= \delta \cos \alpha \Big[\sin \frac{2\pi}{\lambda}(Vt - r)\left(\cos \frac{2\pi}{\lambda}\Delta r - 1\right) \\ &\qquad - \cos \frac{2\pi}{\lambda}(Vt - r) \sin \frac{2\pi}{\lambda}\Delta r \Big], \\ \Delta\eta &= \delta \cos \beta \Big[\sin \frac{2\pi}{\lambda}(Vt - r)\left(\cos \frac{2\pi}{\lambda}\Delta r - 1\right) \\ &\qquad - \cos \frac{2\pi}{\lambda}(Vt - r) \sin \frac{2\pi}{\lambda}\Delta r \Big], \\ \Delta\zeta &= \delta \cos \gamma \Big[\sin \frac{2\pi}{\lambda}(Vt - r)\left(\cos \frac{2\pi}{\lambda}\Delta r - 1\right) \\ &\qquad - \cos \frac{2\pi}{\lambda}(Vt - r) \sin \frac{2\pi}{\lambda}\Delta r \Big]. \end{aligned}\right\} \tag{105}$$

Substituting these values in Eqs. (18), and, since the medium is homogeneous, the sums arising from the substitution of the second part of the values of $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, and which are of the form

$$\left.\begin{aligned}
&\Sigma\mu\, \phi(r) \sin\frac{2\pi}{\lambda}\Delta r,\\
&\Sigma\mu\, \psi(r)\frac{\Delta y^2}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\
&\Sigma\mu\, \psi(r)\frac{\Delta y\,\Delta z}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\
&\Sigma\mu\, \psi(r)\frac{\Delta x\,\Delta y}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\
&\Sigma\mu\, \psi(r)\frac{\Delta x^2}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\
&\Sigma\mu\, \psi(r)\frac{\Delta z^2}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\
&\Sigma\mu\, \psi(r)\frac{\Delta x\,\Delta z}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,
\end{aligned}\right\} \tag{106}$$

all reduce to zero, because they are formed of terms which, two and two, are equal, with contrary signs; for, to the values of Δx, Δy, Δz, equal, with contrary signs, correspond values of Δr which are also equal and have contrary signs. Then, replacing $\cos\frac{2\pi}{\lambda}\Delta r$ by its equal, $1 - 2\sin^2\frac{\pi}{\lambda}\Delta r$, and $\delta\sin\frac{2\pi}{\lambda}(Vt - r)$ by its equal ε, Eqs. (18) become, for plane waves,

$$X_1 = -\frac{X}{2m} = \cos\alpha\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta r$$
$$+ \cos\beta\,\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta y}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r + \cos\gamma\,\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta z}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r,$$

$$Y_1 = -\frac{Y}{2m} = \cos\alpha\,\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta y}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r$$
$$+ \cos\beta\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta r$$
$$+ \cos\gamma\,\Sigma\mu\,\psi(r)\frac{\Delta y\,\Delta z}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r,$$

$$Z_1 = -\frac{Z}{2m} = \cos\alpha\, \Sigma\mu\, \psi(r) \frac{\Delta x\, \Delta z}{r^2} \sin^2\frac{\pi}{\lambda}\Delta r$$

$$+ \cos\beta\, \Sigma\mu\, \psi(r) \frac{\Delta y\, \Delta z}{r^2} \sin^2\frac{\pi}{\lambda}\Delta r$$

$$+ \cos\gamma\, \Sigma\mu \left[\varphi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta r. \tag{107}$$

91. The conditions for the propagation of the plane wave without change are

$$\frac{X_1}{\cos\alpha} = \frac{Y_1}{\cos\beta} = \frac{Z_1}{\cos\gamma} = U_1. \tag{108}$$

Substituting, in Eq. (107), for Δr its equal,

$$\Delta x \cos l + \Delta y \cos m + \Delta z \cos n = \Delta r, \tag{109}$$

and substituting in Eqs. (108) the values of X_1, Y_1, Z_1, thus obtained, we will have two relations which, with

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \tag{110}$$

will enable us to determine the angles α, β, γ, which the displacement should make with the axes, in order that the propagation of the plane wave may be possible.

92. Because of the equality of the coefficients of $\cos\beta$ and $\cos\alpha$ in the first and second of Eqs. (107), and of $\cos\beta$ and $\cos\gamma$ in the third and second, and of $\cos\alpha$ and $\cos\gamma$ in the third and first, we can, by substitutions and reductions similar to those employed in Art. 33, deduce corresponding principles, and hence determine that, for each direction of the plane wave, there correspond, for the molecular displacements, three rectangular directions such that the plane wave may be propagated without change, and that these three directions are parallel to the three axes of an ellipsoid whose equation is,

$$\left.\begin{aligned}
&x^2\,\Sigma\mu\left[\phi(r)+\psi(r)\frac{\Delta x^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta r\\
&+y^2\,\Sigma\mu\left[\phi(r)+\psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta r\\
&+z^2\,\Sigma\mu\left[\phi(r)+\psi(r)\frac{\Delta z^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta r\\
&+2xy\,\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta y}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r\\
&+2xz\,\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta z}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r\\
&+2yz\,\Sigma\mu\,\psi(r)\frac{\Delta y\,\Delta z}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r
\end{aligned}\right\}=1. \qquad (111)$$

This is called either the *inverse ellipsoid* or the *ellipsoid of polarization.* Having also the relation expressed in Eq. (109), we see that the coefficients of Eq. (111) depend upon the angles l, m, n, which determine the direction of the plane wave, upon certain constants which define the constitution of the medium, and upon the wave length. The velocity of propagation is inversely proportional to the length of that axis of the ellipsoid to which the molecular displacements are parallel.

93. ***Relation between the Velocity of Wave Propagation of Plane Waves and the Wave Length in Isotropic Media.*** All directions being identical in isotropic media, we will assume the plane wave normal to the axis of x. We then have $\Delta r = \Delta x$, and

$$\left.\begin{aligned}
&\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta y}{r^2}\sin^2\frac{\pi}{\lambda}\Delta x=0,\\
&\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta z}{r^2}\sin^2\frac{\pi}{\lambda}\Delta x=0,\\
&\Sigma\mu\,\psi(r)\frac{\Delta y\,\Delta z}{r^2}\sin^2\frac{\pi}{\lambda}\Delta x=0;
\end{aligned}\right\} \qquad (112)$$

and Eqs. (107) reduce to

$$\left.\begin{aligned} X_1 &= \cos\alpha\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x, \\ Y_1 &= \cos\beta\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x, \\ Z_1 &= \cos\gamma\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x; \end{aligned}\right\} \quad (113)$$

and the equation of the ellipsoid to

$$\left.\begin{aligned} & x^2\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x \\ + & y^2\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x \\ + & z^2\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x \end{aligned}\right\} = 1; \quad (114)$$

and, since all directions perpendicular to the axis of x are identical with reference to the plane of the wave, we have $\Delta y = \Delta z$, and Eq. (114) of the ellipsoid becomes one of revolution about the axis of x. Whence, we conclude that, in an isotropic medium, a plane wave normal to a given direction can be propagated without change, whenever the molecular displacement is parallel or perpendicular to this direction. To any one direction of normal propagation in such a medium, there corresponds an infinite number of waves with transversal vibrations, having the same velocity, and but *one* wave with longitudinal vibrations whose velocity is different from those with transversal vibrations.

94. For the wave with longitudinal vibrations, we have

$$\alpha = 0, \qquad \beta = \gamma = 90^\circ,$$

$$\left.\begin{aligned} U_1 = X_1 &= \Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x, \\ Y_1 &= Z_1 = 0; \end{aligned}\right\} \quad (115)$$

and, from Eqs. (96) and (107), we have

$$V^2 = \frac{\lambda^2}{2\pi^2}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x. \quad (116)$$

In Acoustics, it will be shown that sound is due to longitudinal vibrations of the medium. This equation will then be applicable in all cases of sound arising from such vibrations, and will be referred to, in that branch of the subject. In Optics, it will be shown that transversal vibrations only are efficacious in producing light.

95. For waves with transversal vibrations in isotropic media, the velocity is independent of the direction of the displacement. We can then suppose the displacement parallel to the axis of y, and thus have

$$\alpha = \gamma = 90^\circ, \qquad \beta = 0,$$

$$\left.\begin{aligned} X_1 &= Z_1 = 0, \\ Y_1 &= \Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x = U_1; \end{aligned}\right\} \quad (117)$$

and $$V^2 = \frac{\lambda^2}{2\pi^2}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x. \quad (118)$$

This equation is applicable in light, for the determination of wave velocity in isotropic and homogeneous media, and will be used hereafter in determining the velocity of light propagation.

By developing $\sin^2\frac{\pi}{\lambda}\Delta x$ into a series, we find

$$\sin^2\frac{\pi}{\lambda}\Delta x = \frac{\pi^2}{\lambda^2}\Delta x^2 - \frac{1}{3}\cdot\frac{\pi^4}{\lambda^4}\Delta x^4 + \frac{2}{45}\cdot\frac{\pi^6}{\lambda^6}\Delta x^6 - \frac{1}{315}\cdot\frac{\pi^8}{\lambda^8}\Delta x^8 + \text{etc.}$$

Substituting this in Eq. (118), we obtain

$$V^2 = a + \frac{b}{\lambda^2} + \frac{c}{\lambda^4} + \frac{d}{\lambda^6} + \text{etc.}, \quad (119)$$

in which $a, b, c, \ldots.$ have for values,

$$\left.\begin{aligned} a &= \frac{1}{2}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta x^2, \\ b &= -\frac{\pi^2}{6}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta x^4, \\ c &= \frac{\pi^4}{45}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta x^6, \\ d &= - \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (120)$$

These constants depend only on the *constitution* of the medium, and decrease very rapidly in value, for Δx is always a very small quantity. If the wave length be not excessively small, if it surpasses a certain value which observation only can determine, the terms of the second member of Eq. (119) will have very rapidly decreasing values, and we will obtain an expression approximately near to V^2 by taking only the first few terms. Hence, a must be positive, and, since observation shows that the most refrangible rays are those of the shortest wave length, and that, as a consequence, V decreases with λ, b is necessarily negative.

96. Hence, in isotropic media, the elasticity being uniform in all directions, the form of the wave surface will be spherical, and when the displacements are longitudinal, its radius at the unit time from the epoch will be the value of V obtained from Eq. (116); when the displacements are transversal, the radius will be the value of V in Eq. (118). The former relates wholly to waves of sound, and the latter to those of light.

The subsequent discussion will now apply to transversal vibrations alone, and the conclusions derived belong therefore to the transmission of light undulations.

Experiment shows that the media which transmit the waves of light are not in general isotropic, and as a consequence the form of the wave surface will not be spherical. We will, therefore, now seek the form of this surface in the general case, and make use of the properties of plane waves for this purpose.

97. ***Plane Waves in a Homogeneous Medium of Three Unequal Elasticities in Rectangular Directions.*** In the plane wave, the following conclusions have been deduced:

1°. The displacements of the molecules, in each position of the same plane wave, must be rectilineal and parallel to each other and to their original directions.

2°. The elastic forces developed by these displacements must be either in the directions of the displacements or alone efficacious in these directions.

3°. The propagation of the plane wave unaltered is then possible.

These conclusions involve, as consequences, a constancy of velocity of propagation when the plane wave is unchanged in direction, and a variation in the velocity as the direction is changed. Hence, if the elasticities of a homogeneous medium differ in all directions, and we suppose plane waves, having all possible positions, originate at any point m of an indefinite medium, these plane waves, at the end of a unit of time, will be at different distances from m. The surface which is the envelope of all these plane waves at this instant is called the *wave surface.*

98. Let $$a > b > c$$

be the principal axes of elasticity of such a medium. Then a^2, b^2, c^2, will measure the *elastic forces* developed in these directions by a displacement equal to unity, and any of the surfaces of elasticity heretofore determined can be used to obtain the elastic forces developed by an equal displacement in the direction of the corresponding radius vector of the surface. The velocity of wave propagation being proportional to the square root of the elastic force, Eq. (96), its value can be found when the elastic force due to the displacement in any direction is known.

99. Fresnel made use of the single-napped surface of elasticity whose equation is

$$a^2x^2 + b^2y^2 + c^2z^2 = r^4; \qquad (121)$$

but for plane waves, the inverse ellipsoid of elasticity or first ellipsoid,

$$a^2x^2 + b^2y^2 + c^2z^2 = 1, \qquad (E)$$

together with its reciprocal ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad (W)$$

can be more readily used, because of its better known properties. The squares of the semi-axes of (W) and of the reciprocals of (E) are the principal elasticities of the medium.

100. There are two cases to consider:

1°. *The plane of the wave contains two of the principal axes,* and hence is one of the *principal planes* of the medium. The plane cuts the ellipsoids in ellipses whose semi-axes are either two of the principal axes. Whatever be the direction and amount of the displacement, it may be replaced by its components in the direction of the axes proportional to cos α and sin α, α being the angle made by the displacement with either axis.

Considering these separately, we see: 1°, that each will communicate to the molecules in the adjacent plane analogous rectilineal motions which will be propagated without alteration of direction; 2°, that the elasticities, and hence the velocities of propagation which belong to these two, are different, and that after a time there will be two series of molecules situated in parallel planes, parallel also to the primitive plane, which will contain all of the original energy; 3°, that the vibrations of the molecules in these two plane waves will be at right angles to each other.

101. 2°. *The plane wave is any whatever.* The sections of the ellipsoids will be ellipses, but will not in general contain either of the axes of the ellipsoids. There will then be no direction of the displacement that can give a resultant elastic force in the direction of the displacement. It is, therefore, essential for a rectilineal oscillation of the molecule and for a consecutive transmission of this oscillation, that there should be no tendency of the rectilineal displacement to be deflected on either side, but that the line of the resultant force should be projected upon the displacement. As it is not in general in the plane, but oblique to it, it can be resolved into two components: one normal to the plane, which is not effective in light undulations; and the other, which is alone efficacious, in the direction of the displacement. In each elliptical section there are two such directions, which are named *singular directions,* and which are perpendicular to each other. Assume any plane section through the centre of (E); the elasticity measured by the squares of the reciprocals of the radii-vectores is the same to the right and left for the two axes of the section, and is the same only for them. Through either of the axes pass the normal plane to the section; it will cut all the parallel plane sections in their homologous axes. With reference to this normal plane, the radii-vectores, and therefore the elasticities of each section, are symmetrical. Hence, if the displacement be along one of the axes of the section, the total elastic force will be in the normal plane, and will be projected on the axis of the section. And since the ellipsoid semidiameters are inversely as the velocities of propagation, the reciprocals of the axes will measure the velocities of wave propagation. Hence is established the fact that for each section there are two of these singular directions, and that they are rectangular. These two singular directions perform the same function for the vibrations of the plane wave as do the axes of elasticity themselves when the plane wave contains them. Each vibration is replaced by two others in the direction of the singular directions, and these two components proceed in the medium without change of direction, but with different velocities, so that there are then, in the general case, two plane waves parallel to each other and to the original plane wave. If α be the angle made by the displacement with one of the axes, the component displacements will be proportional to $\cos \alpha$ and $\sin \alpha$, and the elastic intensities to $\cos^2 \alpha$ and $\sin^2 \alpha$. Whatever may be the direction of the original supposed vibration in the plane wave, the two plane waves which replace it are always the two above designated.

102. If the plane of the wave coincides with either of the circular sections of the ellipsoid, the plane wave will be propagated without alteration, whatever be the direction of the displacement, with a velocity equal to b, the reciprocal of the mean semi-axis of the ellipsoid.

103. ***The Double-Napped Surface of Elasticity.*** If through the centre of (E) we pass any plane, and on the normal to the section at the centre set off distances inversely proportional to the semi-axes of the section, the locus of all these pairs of points is called the *double-napped surface of elasticity*. For, each radius vector measures the velocity of propagation of one of the plane waves, arising from a displacement in the plane of section, and the square of each of these normal velocities is the measure of the elastic force developed by the component displacement along the axes of the section.

104. If through each of the points so determined planes be passed parallel to the corresponding plane of section, the envelope of all these planes will be, by definition, the wave surface. Hence, the latter can be constructed by points from this surface of elasticity.

105. To get the polar equation of the latter surface, let us take for co-ordinate axes the principal axes of the medium ; let l, m, n, be the angles made by the normal to the plane wave with these axes, x, y, z, respectively; α, β, γ, those which one of the axes of the ellipse of section make with the same axes; then we have

$$\cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n = 0. \qquad (122)$$

The elastic force developed by a displacement parallel to the axis of section is projected on the plane of the wave parallel to this displacement, and its components are

$$X = a^2\cos\alpha, \quad Y = b^2\cos\beta, \quad Z = c^2\cos\gamma. \qquad (123)$$

The cosines of the angles which this elastic force makes with the axes are then proportional to these values. An auxiliary right line perpendicular to the direction of the elastic force and to the displacement will lie in the plane of the wave, and if u, v, w, be the angles which it makes with the axes, we will have

$$\left.\begin{aligned} a^2\cos\alpha\cos u + b^2\cos\beta\cos v + c^2\cos\gamma\cos w &= 0, \\ \cos\alpha\cos u + \cos\beta\cos v + \cos\gamma\cos w &= 0, \\ \cos l\cos u + \cos m\cos v + \cos n\cos w &= 0. \end{aligned}\right\} \qquad (124)$$

Representing the velocity of propagation of the plane wave by V, we have

$$V^2 = a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma. \qquad (125)$$

Combining the above equations, and eliminating the quantities α, β, γ, u, v, w, we will have an equation containing V, l, m, n, which will be that of the surface required. To eliminate u, v, w, we will make use of the method of indeterminate coefficients; thus, multiply Eqs. (124) by B, A, and unity, respectively, add the three resulting equations, and from the conditions for B and A that the coefficients of $\cos v$ and $\cos w$ shall reduce to zero, we will have

$$\left.\begin{aligned} (A + Ba^2)\cos\alpha + \cos l &= 0, \\ (A + Bb^2)\cos\beta + \cos m &= 0, \\ (A + Bc^2)\cos\gamma + \cos n &= 0. \end{aligned}\right\} \qquad (126)$$

Multiply these by $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, respectively, add, and reduce by Eqs. (122), (125); we will have

$$A + BV^2 = 0. \qquad (127)$$

Substitute this value of A in Eqs. (126), and we have

$$\left.\begin{aligned} \cos l &= B\,(V^2 - a^2)\cos\alpha, \\ \cos m &= B\,(V^2 - b^2)\cos\beta, \\ \cos n &= B\,(V^2 - c^2)\cos\gamma. \end{aligned}\right\} \qquad (128)$$

From which we get

$$\left.\begin{aligned} \frac{\dfrac{\cos l}{V^2 - a^2}}{\cos\alpha} = \frac{\dfrac{\cos m}{V^2 - b^2}}{\cos\beta} = \frac{\dfrac{\cos n}{V^2 - c^2}}{\cos\gamma} \\ = \sqrt{\frac{\cos^2 l}{(V^2 - a^2)^2} + \frac{\cos^2 m}{(V^2 - b^2)^2} + \frac{\cos^2 n}{(V^2 - c^2)^2}}. \end{aligned}\right\} \qquad (129)$$

Replacing $\cos\alpha$, $\cos\beta$, $\cos\gamma$, in Eq. (122), by their proportional quantities, $\dfrac{\cos l}{V^2 - a^2}$, $\dfrac{\cos m}{V^2 - b^2}$, $\dfrac{\cos n}{V^2 - c^2}$, we have

$$\frac{\cos^2 l}{V^2 - a^2} + \frac{\cos^2 m}{V^2 - b^2} + \frac{\cos^2 n}{V^2 - c^2} = 0, \qquad (130)$$

the polar equation of the double-napped surface of elasticity, in which V is any radius vector.

106. ***The Wave Surface.*** Through any point of the surface of elasticity pass a plane perpendicular to the radius vector at that point, and let r, λ, μ, ν, be the polar co-ordinates of any point of the plane. The equation of the plane will be

$$\cos l \cos\lambda + \cos m \cos\mu + \cos n \cos\nu = \frac{V}{r}. \qquad (131)$$

We have also, as equations of condition,

$$\cos^2 l + \cos^2 m + \cos^2 n = 1, \qquad (132)$$

$$\frac{\cos^2 l}{V^2 - a^2} + \frac{\cos^2 m}{V^2 - b^2} + \frac{\cos^2 n}{V^2 - c^2} = 0. \qquad (133)$$

The wave surface is the enveloping surface of the planes given by Eq. (131), and its equation can be determined by eliminating V, l, m, n, and finding an equation between r, λ, μ, ν. To do this, differentiate Eqs. (131), (132), (133), regarding $\cos l$ and $\cos m$ as independent variables, and we will have

$$\left.\begin{aligned}
&\cos\lambda + \cos\nu\,\frac{d\cos n}{d\cos l} = \frac{1}{r}\cdot\frac{dV}{d\cos l},\\
&\cos l + \cos n\,\frac{d\cos n}{d\cos l} = 0,\\
&\frac{\cos l}{V^2 - a^2} + \frac{\cos n}{V^2 - c^2}\cdot\frac{d\cos n}{d\cos l}\\
&= \frac{dV}{d\cos l}\,V\left[\frac{\cos^2 l}{(V^2 - a^2)^2} + \frac{\cos^2 m}{(V^2 - b^2)^2} + \frac{\cos^2 n}{(V^2 - c^2)^2}\right];
\end{aligned}\right\} \quad (134)$$

$$\left.\begin{aligned}
&\cos\mu + \cos\nu\,\frac{d\cos n}{d\cos m} = \frac{1}{r}\cdot\frac{dV}{d\cos m},\\
&\cos m + \cos n\,\frac{d\cos n}{d\cos m} = 0,\\
&\frac{\cos m}{V^2 - b^2} + \frac{\cos n}{V^2 - c^2}\cdot\frac{d\cos n}{d\cos m}\\
&= \frac{dV}{d\cos m}\,V\left[\frac{\cos^2 l}{(V^2 - a^2)^2} + \frac{\cos^2 m}{(V^2 - b^2)^2} + \frac{\cos^2 n}{(V^2 - c^2)^2}\right].
\end{aligned}\right\} \quad (135)$$

107. To eliminate $\frac{d\cos n}{d\cos l}$, $\frac{d\cos n}{d\cos m}$, $\frac{dV}{d\cos l}$, $\frac{dV}{d\cos m}$, multiply Eqs. (134) by 1, A, and $-B$, respectively, add the resulting equations together, and perform the same operations on Eqs. (135). Supposing the indeterminate quantities A and B to have such values as will make the coefficients of $\frac{d\cos n}{d\cos l}$, $\frac{d\cos n}{d\cos m}$, $\frac{dV}{d\cos l}$, $\frac{dV}{d\cos m}$, equal to zero, we will have

$$\left.\begin{aligned}
&\cos\lambda + A\cos l = B\,\frac{\cos l}{V^2 - a^2},\\
&\cos\mu + A\cos m = B\,\frac{\cos m}{V^2 - b^2},\\
&\cos\nu + A\cos n = B\,\frac{\cos n}{V^2 - c^2},\\
&\frac{1}{r} = BV\left[\frac{\cos^2 l}{(V^2 - a^2)^2} + \frac{\cos^2 m}{(V^2 - b^2)^2} + \frac{\cos^2 n}{(V^2 - c^2)^2}\right].
\end{aligned}\right\} \quad (136)$$

Multiply the first three of the above equations by $\cos l$, $\cos m$, and $\cos n$, respectively, add the resulting equations, and reduce by Eqs. (131), (132), (133); we will have

$$A + \frac{V}{r} = 0. \tag{137}$$

Squaring the first three of Eqs. (136), and adding, we get, after reduction

$$1 + 2A\frac{V}{r} + A^2 = \frac{B}{rV}; \tag{138}$$

whence, we have

$$A = -\frac{V}{r}, \qquad B = \frac{V}{r}(r^2 - V^2). \tag{139}$$

Substituting these values in Eqs. (136), we obtain

$$\left.\begin{aligned} \frac{r\cos\lambda}{r^2 - a^2} &= \frac{V\cos l}{V^2 - a^2}, \\ \frac{r\cos\mu}{r^2 - b^2} &= \frac{V\cos m}{V^2 - b^2}, \\ \frac{r\cos\nu}{r^2 - c^2} &= \frac{V\cos n}{V^2 - c^2}, \\ \frac{1}{r^2 - V^2} &= V^2\left[\frac{\cos^2 l}{(V^2 - a^2)^2} + \frac{\cos^2 m}{(V^2 - b^2)^2} + \frac{\cos^2 n}{(V^2 - c^2)^2}\right] \\ &= r^2\left[\frac{\cos^2\lambda}{(r^2 - a^2)^2} + \frac{\cos^2\mu}{(r^2 - b^2)^2} + \frac{\cos^2\nu}{(r^2 - c^2)^2}\right]. \end{aligned}\right\} \tag{140}$$

The first three equations (140) can be placed under the form,

$$\left.\begin{aligned} \cos\lambda - \frac{V}{r}\cos l &= (r^2 - V^2)\frac{\cos\lambda}{r^2 - a^2}, \\ \cos\mu - \frac{V}{r}\cos m &= (r^2 - V^2)\frac{\cos\mu}{r^2 - b^2}, \\ \cos\nu - \frac{V}{r}\cos n &= (r^2 - V^2)\frac{\cos\nu}{r^2 - c^2}. \end{aligned}\right\} \tag{141}$$

Adding these, after multiplying them by $\cos\lambda$, $\cos\mu$, $\cos\nu$, respectively, and reducing by Eq. (131), we have

$$1 - \frac{V^2}{r^2} = (r^2 - V^2)\left(\frac{\cos^2 \lambda}{r^2 - a^2} + \frac{\cos^2 \mu}{r^2 - b^2} + \frac{\cos^2 \nu}{r^2 - c^2}\right); \quad (142)$$

whence, dividing by $r^2 - V^2$, we have

$$\frac{\cos^2 \lambda}{r^2 - a^2} + \frac{\cos^2 \mu}{r^2 - b^2} + \frac{\cos^2 \nu}{r^2 - c^2} = \frac{1}{r^2}, \quad (143)$$

the polar equation of the wave surface.

108. A more advantageous form for discussion can be obtained by subtracting the identical equation,

$$\frac{1}{r^2} = \frac{\cos^2 \lambda}{r^2} + \frac{\cos^2 \mu}{r^2} + \frac{\cos^2 \nu}{r^2}, \quad (144)$$

from the equation above, by which there results

$$\frac{a^2 \cos^2 \lambda}{r^2 - a^2} + \frac{b^2 \cos^2 \mu}{r^2 - b^2} + \frac{c^2 \cos^2 \nu}{r^2 - c^2} = 0. \quad (145)$$

109. To obtain the equation of the wave surface in rectangular co-ordinates substitute for $\cos \lambda$, $\cos \mu$, $\cos \nu$, and r, their equals, $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$, and $\sqrt{x^2 + y^2 + z^2}$; whence, we have

$$\left.\begin{array}{r}(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)x^2 \\ - b^2(c^2 + a^2)y^2 - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0.\end{array}\right\} \quad (146)$$

This equation being of the fourth degree, the surface is of the fourth order, and, as will be shown hereafter, consists of two distinct nappes, having but four points in common.

110. If two of the velocities become equal, as, for example, $b = c$, the equation gives

$$x^2 + y^2 + z^2 = b^2, \quad (147)$$

$$a^2x^2 + b^2(y^2 + z^2) = a^2b^2, \quad (148)$$

which shows that the wave surface, under this supposition, consists of a spherical surface and that of an ellipsoid of revolution tangent to the sphere at the extremity of its polar axis.

Finally, if the three principal velocities become equal, or $a = b = c$, as in isotropic media, Eq. (146) becomes

$$x^2 + y^2 + z^2 = a^2, \tag{149}$$

and the wave surface becomes spherical, as has been heretofore shown.

111. ***Construction of the Wave Surface by Means of the Ellipsoid* (W).** Let us suppose that the ellipsoid (W),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{150}$$

be cut by any plane through its centre, and that distances be laid off on the normal equal to the semi-axes of the elliptical section. Referring to the construction of the double-napped surface of elasticity by means of the ellipsoid (E),

$$a^2x^2 + b^2y^2 + c^2z^2 = 1, \tag{151}$$

we see that in designating the polar co-ordinates of the points constructed by the aid of (W) by r, λ, μ, ν, the equation of their loci can be obtained from the equation of the double-napped surface,

$$\frac{\cos^2 l}{V^2 - a^2} + \frac{\cos^2 m}{V^2 - b^2} + \frac{\cos^2 n}{V^2 - c^2} = 0, \tag{152}$$

by substituting for V^2, a^2, b^2, c^2, l, m, n, respectively, $\frac{1}{r^2}$, $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, λ, μ, ν. We thus obtain

$$\frac{\cos^2 \lambda}{\frac{1}{r^2} - \frac{1}{a^2}} + \frac{\cos^2 \mu}{\frac{1}{r^2} - \frac{1}{b^2}} + \frac{\cos^2 \nu}{\frac{1}{r^2} - \frac{1}{c^2}} = 0, \tag{153}$$

or

$$\frac{a^2 \cos^2 \lambda}{r^2 - a^2} + \frac{b^2 \cos^2 \mu}{r^2 - b^2} + \frac{c^2 \cos^2 \nu}{r^2 - c^2} = 0, \tag{154}$$

which is the equation of the wave surface. Hence, points of the wave surface can be constructed from the ellipsoid (W) in precisely the same manner as points of the surface of elasticity from the ellipsoid (E), except that in the former, distances *equal* to the semi-axes are laid off on the normal, and in the latter the distances are *equal to the reciprocals* of the semi-axes.

112. *Direction of the Vibration at any Point of the Wave Surface.* Let us consider any plane wave tangent to the wave surface; the displacement propagated by this plane wave makes the angles α, β, γ, with the axes; the radius vector of the wave surface at the point of tangency makes with the axes the angles λ, μ, ν; therefore the angle between these two lines will determine the required direction.

Eliminate in Eq. (129) the angles l, m, n, by means of the first three of Eqs. (140), which, with the last of Eqs. (136), will give

$$\left.\begin{aligned}\frac{\frac{\cos\lambda}{r^2-a^2}}{\cos\alpha} = \frac{\frac{\cos\mu}{r^2-b^2}}{\cos\beta} = \frac{\frac{\cos\nu}{r^2-c^2}}{\cos\gamma} \\ = \sqrt{\frac{\cos^2\lambda}{(r^2-a^2)^2} + \frac{\cos^2\mu}{(r^2-b^2)^2} + \frac{\cos^2\nu}{(r^2-c^2)^2}} \\ = \frac{V}{r}\sqrt{\frac{\cos^2 l}{(V^2-a^2)^2} + \frac{\cos^2 m}{(V^2-b^2)^2} + \frac{\cos^2 n}{(V^2-c^2)^2}} \\ = \frac{V}{r}\sqrt{\frac{1}{BVr}} = \frac{1}{r\sqrt{r^2-V^2}};\end{aligned}\right\} \quad (155)$$

whence,

$$\left.\begin{aligned}\frac{\cos\lambda}{r^2-a^2} = \frac{\cos\alpha}{r\sqrt{r^2-V^2}}, \\ \frac{\cos\mu}{r^2-b^2} = \frac{\cos\beta}{r\sqrt{r^2-V^2}}, \\ \frac{\cos\nu}{r^2-c^2} = \frac{\cos\gamma}{r\sqrt{r^2-V^2}},\end{aligned}\right\} \quad (156)$$

Substituting in Eq. (143) of the wave surface, we have

$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu = \sqrt{1-\frac{V^2}{r^2}}. \quad (157)$$

In the figure, let M be any point of the wave surface, OM the radius vector, and OP the perpendicular V on the tangent plane to the wave surface; $\frac{V}{r}$ is

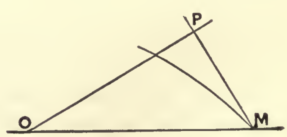


Figure 16.

then the cosine of POM, and $\sqrt{1 - \frac{V^2}{r^2}}$ is its sine; hence, OMP is complementary to POM, and therefore the vibrations at M are directed along the line PM. We conclude, therefore, that the direction of the vibrations of the molecule at any point of the wave surface is along the projection of the radius vector on the tangent plane at that point.

When the tangent plane is normal to the radius vector, as is the case at the extremities of the axes, this determination is not applicable, but the direction is in these cases easily found. The plane OMP, which contains the radius vector and the direction of the corresponding vibration, is called the *plane of vibration.*

113. ***Relations between the Directions of Normal Propagation of Plane Waves, the Directions of Radii-Vectores of the Wave Surface, and the Directions of Vibrations.*** By the preceding theorem we have seen that, in any plane wave whatever, the normal to this plane, the direction of the vibrations in this wave, and the radius vector drawn to the wave surface at the point of tangency, are all contained in the same plane. Besides, for each normal direction of propagation of a plane wave, there correspond for the vibrations, two directions parallel to the axes of the elliptical section of the ellipsoid (E) made by a parallel plane. These directions, therefore, being perpendicular to each other, the planes which contain, at the same time, the same direction of normal propagation, the two vibrations, and the two corresponding radii-vectores, are rectangular.

114. Since the wave surface has two nappes, each radius vector will give two directions for the vibrations. We will now show that the planes which contain a radius vector and the directions of the two corresponding vibrations are also rectangular; and for this purpose we shall show that the two vibrations which correspond to the same radius vector are contained in the two planes passing through this radius vector and the axes of the elliptical section, that a plane perpendicular to the radius vector cuts out of the ellipsoid (W).

115. Let ϕ, ψ, χ, be the angles made by one of the axes of this elliptical section with the co-ordinate axes; it is then necessary to demonstrate that the three lines (α, β, γ), (λ, μ, ν), (ϕ, ψ, χ), are all in the same plane.

Eq. (129), which gives the relations existing between the angles α, β, γ, made by one of the axes of ellipsoid (E) with the co-ordinate axes, and the right line l, m, n, perpendicular to the elliptical section, can be applied to the analogous case of the elliptical section of (W) and the normal radius vector, by replacing in this equation V^2, α, β, γ, l, m, n, a^2, b^2, c^2, by $\frac{1}{r^2}$, ϕ, ψ, χ, λ, μ, ν, $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, respectively, which will give

$$\left.\begin{aligned} \frac{\frac{a^2 \cos \lambda}{r^2 - a^2}}{\cos \phi} = \frac{\frac{b^2 \cos \mu}{r^2 - b^2}}{\cos \psi} = \frac{\frac{c^2 \cos \nu}{r^2 - c^2}}{\cos \chi} \\ = \sqrt{\frac{a^4 \cos^2 \lambda}{(r^2 - a^2)^2} + \frac{b^4 \cos^2 \mu}{(r^2 - b^2)^2} + \frac{c^4 \cos^2 \nu}{(r^2 - c^2)^2}}. \end{aligned}\right\} \quad (158)$$

Let the auxiliary right line defined by the angles (A, B, C) be drawn perpendicular to the two right lines (λ, μ, ν) and (ϕ, ψ, χ); we will have

$$\left.\begin{aligned} \cos A \cos \lambda + \cos B \cos \mu + \cos C \cos \nu = 0, \\ \cos A \cos \phi + \cos B \cos \psi + \cos C \cos \chi = 0. \end{aligned}\right\} \quad (159)$$

Replacing, in the last equation, for $\cos \phi$, $\cos \psi$, $\cos \chi$, the quantities proportional to them in Eq. (158), we have

$$\frac{a^2 \cos \lambda}{r^2 - a^2} \cos A + \frac{b^2 \cos \mu}{r^2 - b^2} \cos B + \frac{c^2 \cos \nu}{r^2 - c^2} \cos C = 0. \quad (160)$$

Adding this to the first of Eqs. (159), we have

$$\frac{\cos \lambda}{r^2 - a^2} \cos A + \frac{\cos \mu}{r^2 - b^2} \cos B + \frac{\cos \nu}{r^2 - c^2} \cos C = 0; \quad (161)$$

and recollecting that the relations

$$\frac{\frac{\cos \lambda}{r^2 - a^2}}{\cos \alpha} = \frac{\frac{\cos \mu}{r^2 - b^2}}{\cos \beta} = \frac{\frac{\cos \nu}{r^2 - c^2}}{\cos \gamma} \quad (162)$$

exist, we have finally

$$\cos \alpha \cos A + \cos \beta \cos B + \cos \gamma \cos C = 0. \quad (163)$$

Hence, the three right lines (α, β, γ), (λ, μ, ν), (ϕ, ψ, χ), being perpendicular to the right line (A, B, C), are all contained in the same plane, and therefore we conclude that the planes which contain, at the same time, the same radius vector, the two vibrations, and the two corresponding directions of the normal propagation, are rectangular.

116. ***Discussion of the Wave Surface.*** Resuming Eq. (146),

$$(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)x^2 - b^2(c^2 + a^2)y^2 - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0,$$

and making in succession $x = 0$, $y = 0$, $z = 0$, we get for the sections made by the co-ordinate planes yz, xz, and xy, respectively,

$$(y^2 + z^2 - a^2)(b^2y^2 + c^2z^2 - b^2c^2) = 0, \tag{164}$$

$$(x^2 + z^2 - b^2)(a^2x^2 + c^2z^2 - a^2c^2) = 0, \tag{165}$$

$$(x^2 + y^2 - c^2)(a^2x^2 + b^2y^2 - a^2b^2) = 0. \tag{166}$$

Remembering that $a > b > c$, we see that the section in the plane yz will be a circle whose radius is a, entirely outside of an ellipse whose semi-axes are b and c. The section in the plane xy will be a circle with radius c, entirely within the ellipse whose semi-axes are a and b. That in the plane xz will be a circle with radius

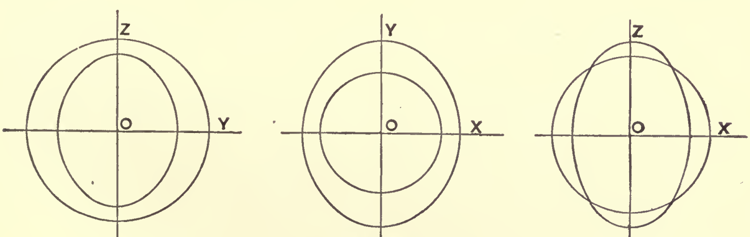


Figure 17.

b, intersecting at four points the ellipse whose semi-axes are a and c. The axis of x pierces the surface at distances equal to $\pm b$ and $\pm c$ from the centre, that of y at distances of $\pm a$ and $\pm c$, and that of z at $\pm a$ and $\pm b$.

117. The surface of elasticity of two nappes cuts the axes in the same points. These principal axes of elasticity have in turn represented the square roots of the elastic forces developed along the three axes of elasticity, the principal velocities of wave propagation, the axes of the ellipsoids, and now serve to fix points on the surface of elasticity and on the wave surface.

118. Let Eq. (146) be represented by $L = 0$, and the angles which a tangent plane to the surface at any point makes with the co-ordinate planes xy, xz, yz, by A, B, C, respectively; then we will have

$$\cos A = \frac{1}{\omega}\cdot\frac{dL}{dz}, \quad \cos B = \frac{1}{\omega}\cdot\frac{dL}{dy}, \quad \cos C = \frac{1}{\omega}\cdot\frac{dL}{dx}, \tag{167}$$

in which,

$$\frac{1}{\omega} = \frac{1}{\sqrt{\left(\frac{dL}{dx}\right)^2 + \left(\frac{dL}{dy}\right)^2 + \left(\frac{dL}{dz}\right)^2}}. \tag{168}$$

Taking the differential coefficients of L with respect to x, y, z, we will have

$$\left.\begin{aligned} \frac{dL}{dx} &= 2x\,(a^2x^2 + b^2y^2 + c^2z^2) + 2a^2x\,(x^2+y^2+z^2-b^2-c^2), \\ \frac{dL}{dy} &= 2y\,(a^2x^2 + b^2y^2 + c^2z^2) + 2b^2y\,(x^2+y^2+z^2-a^2-c^2), \\ \frac{dL}{dz} &= 2z\,(a^2x^2 + b^2y^2 + c^2z^2) + 2c^2z\,(x^2+y^2+z^2-a^2-b^2). \end{aligned}\right\} \tag{169}$$

For $y = 0$, the point of tangency is in the plane xz, and we have

$$\left.\begin{aligned} \frac{dL}{dx} &= 2x\,(a^2x^2 + c^2z^2) + 2a^2x\,(x^2 + z^2 - b^2 - c^2), \\ \frac{dL}{dy} &= 0, \\ \frac{dL}{dz} &= 2z\,(a^2x^2 + c^2z^2) + 2c^2z\,(x^2 + z^2 - a^2 - b^2), \end{aligned}\right\} \tag{170}$$

the second equation showing that the tangent plane is normal to the plane xz.

For $y = 0$, the equation of the surface gives

$$x^2 + z^2 - b^2 = 0, \qquad a^2x^2 + c^2z^2 - a^2c^2 = 0; \tag{171}$$

whence, for the co-ordinates x and z, we have

$$x = \pm c\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \qquad z = \pm a\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}, \qquad (172)$$

which are real so long as $a > b > c$. There are then four points of intersection in the plane xz. Substituting these values in Eqs. (167), we obtain

$$\cos A = \frac{0}{0}, \qquad \cos B = \frac{0}{0}, \qquad \cos C = \frac{0}{0}. \qquad (173)$$

119. The interpretation of these indeterminate values of the cosines is, that at the points considered, a tangent plane to the wave surface may have any position whatever with respect to the co-ordinate planes. This property shows that these points are the vertices of conoidal cusps, each having a tangent cone. These points, called *umbilics*, belong to the exterior and interior nappe of the wave surface, just as the vertex of a cone is common to its upper and lower nappes.

120. The equation of the right lines joining these points, OI, OI′, through the centre in the plane xz is

$$z = \pm \frac{a}{c}\sqrt{\frac{b^2 - c^2}{a^2 - b^2}}\, x, \qquad (174)$$

which shows that the lines are normal to the circular sections of the ellipsoid (W). The lines themselves are called *axes of exterior conical refraction*.

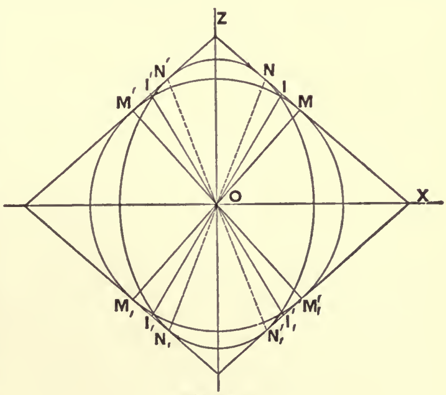


Figure 18.

121. If tangent lines be drawn to the ellipse and circle, as MN, M′N′, they will be parallel to each other, two and two, and symmetrically placed with respect to the axes OX and OZ. The equations of these lines can easily be shown to be

$$z = \pm\sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\, x \pm b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}, \qquad (175)$$

and hence the equation of the line drawn from O perpendicular to the tangent to be

$$z = \mp\sqrt{\frac{b^2 - c^2}{a^2 - b^2}}x, \tag{176}$$

which shows that these lines are normal to the circular sections of the ellipsoid (E).

122. From the properties of this ellipsoid, we see that a plane wave perpendicular to one of the right lines MM_1, $M'M_1'$, and at the same time perpendicular to xz, can be propagated without alteration, whatever may be the direction of the displacement in its plane, and that the velocity of propagation of this plane wave is independent of the direction of the displacement. The lines MM_1, $M'M_1'$, are called the *optic axes* of the medium, or *axes of interior conical refraction.*

123. We see, by comparing Eqs. (174) and (176), that the lines OI and OM differ by the factor $\frac{a}{c}$ in their tangents. This ratio is always very nearly unity, and therefore the lines have nearly the same direction.

124. The planes drawn through the four tangents MN, M'N', etc., perpendicular to the plane xz, are tangent to the wave surface along the circumferences of circles, which are projected in the lines MN, M'N', etc. To show this, let

$$F(x, y, z) = 0 \tag{177}$$

be the equation of the wave surface; then, for points in the plane perpendicular to xz, we have

$$\frac{dF}{dy} = y(a^2x^2 + b^2y^2 + c^2z^2) + b^2y(x^2 + y^2 + z^2) - b^2(a^2 + c^2)y = 0. \tag{178}$$

which can be satisfied by placing

$$y = 0, \tag{179}$$

and $$(a^2 + b^2)x^2 + 2b^2y^2 + (b^2 + c^2)z^2 - b^2(a^2 + c^2) = 0. \tag{180}$$

The first of these equations gives the points of contact in the plane xz; the second represents an ellipsoid. If we combine the equation of the ellipsoid (180) with the equation of the wave surface, eliminating y^2, the resulting equation will be the projection on the plane xz of the intersections of these surfaces, and since the co-ordinates of the points projected satisfy the condition, $\frac{dF}{dy} = 0$, all the points of the wave surface in the tangent plane which is perpendicular to xz, will be obtained from this intersection and projection. The resulting equation after reduction can be put in the form of

$$\left.\begin{aligned} &\left(z + \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x + b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}\right) \\ \times &\left(z - \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x + b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}\right) \\ \times &\left(z + \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x - b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}\right) \\ \times &\left(z - \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x - b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}\right) \end{aligned}\right\} = 0. \qquad (181)$$

This equation can be satisfied by placing each factor separately equal to zero, and each will then be the equation of a plane passed through one of the tangent lines MN, M′N′, M_1N_1, $M_1'N_1'$; hence, each of the four planes touch the surface in those points determined by its intersection with the ellipsoid, and it is readily seen that these curves are the circular sections of the ellipsoid, Eq. (180). The four planes

$$z = \pm x\sqrt{\frac{a^2 - b^2}{b^2 - c^2}} \pm b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}} \qquad (182)$$

are called the *singular tangent planes* of the wave surface.

125. The circles are, in fact, the edges of the conoidal or umbilic cusps, determined by the surface of the tangent cones, reaching their limits by becoming planes in the gradual increase of the inclination of their elements, as the tangential circumference recedes from the cusp points.

It thus appears that the general wave surface consists of two nappes, the one wholly within the other, except at four points, where they unite.

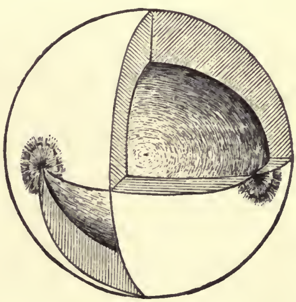


Figure 19.

Fig. 19 represents a model of the wave surface, with sections made by the co-ordinate planes, so as to show the interior nappe.

126. ***Relations between the Velocities and Positions of Plane Waves with respect to the Optic Axes.*** For each direction of normal propagation, two plane waves travel with different velocities, determined, as we have seen, by the equation of the surface of elasticity,

$$\frac{\cos^2 l}{V^2 - a^2} + \frac{\cos^2 m}{V^2 - b^2} + \frac{\cos^2 n}{V^2 - c^2} = 0.$$

This equation can be put under the form

$$\left.\begin{aligned} V^4 - [(b^2+c^2)\cos^2 l + (a^2+c^2)\cos^2 m + (a^2+b^2)\cos^2 n]\,V^2 \\ + b^2c^2\cos^2 l + a^2c^2\cos^2 m + a^2b^2\cos^2 n = 0\,; \end{aligned}\right\} \quad (183)$$

and representing the two square roots by V'^2 and V''^2, we have

$$\left.\begin{aligned} V'^2 + V''^2 = (b^2 + c^2)\cos^2 l + (a^2 + c^2)\cos^2 m \\ + (a^2 + b^2)\cos^2 n, \end{aligned}\right\} \quad (184)$$

$$V'^2 V''^2 = b^2c^2\cos^2 l + a^2c^2\cos^2 m + a^2b^2\cos^2 n. \quad (185)$$

Let θ', θ'', be the angles that the direction of normal propagation makes with the optic axes, and ϕ and $180° - \phi$ the angles that the optic axes make with the axis x, the axis of greatest elasticity, then we have

$$\cos\phi = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \qquad \sin\phi = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}, \quad (186)$$

$$\left.\begin{aligned} \cos\theta' &= \cos\phi\cos l + \sin\phi\cos n, \\ \cos\theta'' &= -\cos\phi\cos l + \sin\phi\cos n; \end{aligned}\right\} \quad (187)$$

whence, we get

$$\cos l = \frac{\cos\theta' - \cos\theta''}{2\cos\phi} = \frac{\cos\theta' - \cos\theta''}{2}\sqrt{\frac{a^2 - c^2}{a^2 - b^2}}, \quad (188)$$

$$\cos n = \frac{\cos\theta' + \cos\theta''}{2\sin\phi} = \frac{\cos\theta' + \cos\theta''}{2}\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}. \quad (189)$$

Substituting these values of $\cos l$ and $\cos n$ in Eqs. (184) and (185), and replacing $\cos^2 m$ by $1 - \cos^2 l - \cos^2 n$, we obtain

$$\left.\begin{aligned} V'^2 + V''^2 &= a^2 + c^2 - \frac{(\cos\theta' - \cos\theta'')^2}{4}(a^2 - c^2) \\ &\qquad + \frac{(\cos\theta' + \cos\theta'')^2}{4}(a^2 - c^2) \\ &= a^2 + c^2 + (a^2 - c^2)\cos\theta'\cos\theta'', \end{aligned}\right\} \quad (190)$$

$$\left.\begin{aligned} V'^2 V''^2 &= a^2c^2 - \frac{(a^2 - c^2)\,c^2}{4}(\cos\theta' - \cos\theta'')^2 \\ &\qquad + \frac{(a^2 - c^2)\,a^2}{4}(\cos\theta' + \cos\theta'')^2 \\ &= a^2c^2 + \frac{(a^2 - c^2)^2}{4}(\cos^2\theta' + \cos^2\theta'') \\ &\qquad + \frac{(a^2-c^2)\,(a^2+c^2)}{2}\cos\theta'\cos\theta''; \end{aligned}\right\} \quad (191)$$

whence,

$$\left.\begin{aligned} (V'^2 - V''^2)^2 &= (V'^2 + V''^2)^2 - 4V'^2V''^2 \\ &= (a^2 + c^2)^2 + (a^2 - c^2)^2\cos^2\theta'\cos^2\theta'' - 4a^2c^2 \\ &\qquad - (a^2 - c^2)^2(\cos^2\theta' + \cos^2\theta'') \\ &= (a^2 - c^2)^2(1 - \cos^2\theta')(1 - \cos^2\theta'') \\ &= (a^2 - c^2)^2\sin^2\theta'\sin^2\theta''; \end{aligned}\right\} \quad (192)$$

and finally, $\quad V'^2 - V''^2 = (a^2 - c^2)\sin\theta'\sin\theta''. \qquad (193)$

This equation establishes the relation between the velocities of the two plane waves which belong to the same direction of normal propagation, and the angles that this direction makes with the optic axes.

127. The directions of the two vibratory motions can be determined by means of the optic axes. These directions are parallel to the axes of the elliptical section of (E) made by the plane normal to the direction of propagation; but the elliptical section is cut by the planes of the two circular sections of the ellipsoid in two equal diameters of the ellipse, since they are equal to the radius b of the circular section; they are therefore equally inclined to the axes of the ellipse. The optical axes being normal to the circular sections, are projected on the plane of the ellipse in two diameters which are perpendicular to those just spoken of, and are therefore also equally inclined to the axes of the ellipse. But these projections are the traces of the planes containing the directions of the normal propagation and each optic axis. We therefore conclude, *that the bisecting planes of the diedral angle formed by the planes containing the direction of any normal propagation and each of the optic axes, are the planes of vibration of the two plane waves corresponding to this normal propagation.*

128. The plane xz being the plane of the optic axes, any direction of normal propagation in this plane will make the diedral angle 0° and 180°, and hence the planes of vibration will be the principal plane xz and a plane containing y and the direction of propagation.

129. ***Relations between the Velocities of Two Rays which are Coincident in Direction, and the Angles that this Direction makes with the Axes of Exterior Conical Refraction.*** The expressions

$$\pm\sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \quad \text{and} \quad \sqrt{\frac{b^2 - c^2}{a^2 - c^2}},$$

being the cosines of the angles that the optic axes make with the axes of x, z, and making use of the analogy existing between the ellipsoid (E) to the surface of elasticity, and the ellipsoid (W) to the wave surface, we will have, by substituting for a^2, b^2, c^2, in the above, $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, the expressions

$$\pm\frac{c}{b}\sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \quad \text{and} \quad \frac{a}{b}\sqrt{\frac{b^2 - c^2}{a^2 - c^2}},$$

for the cosines of the angles that the axes of exterior conical refraction make with the axes of x, z.

If then r and r', the two coincident radii-vectores of the wave surface, represent the ray velocities propagated in the same direction, and u' and u'' be the angles made by this direction with the two axes of exterior conical refraction, a discussion in every way analogous to that above for the optic axes will determine the required relation. This relation may be at once determined by replacing V', V'', θ', θ'', in Eq. (193), by $\frac{1}{r'}$, $\frac{1}{r''}$, u', u'', respectively; we then have

$$\frac{1}{r'^2} - \frac{1}{r''^2} = \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin u' \sin u''. \tag{194}$$

130. The axes of exterior conical refraction being normal to the circular sections of ellipsoid (W), by a similar course of reasoning as in Art. (127), we will arrive at the theorem, that *the bisecting planes of the diedral angle, formed by the planes containing any radius vector of the wave surface and each of the axes of exterior conical refraction, are the planes of vibration of the two rays corresponding to this radius vector.*

131. Thus, from the wave surface we can determine:

1°. The position of the refracted plane waves by its tangent planes.

2°. The direction of the two corresponding rays by the points of contact of the two parallel tangent planes.

3°. The velocities of the two rays by the lengths of the radii-vectores drawn to the points of contact.

4°. The velocities of the two plane waves by the normals from the centre upon the tangent planes.

5°. The interior directions of the molecular vibrations by the projection of the radii-vectores on the tangent planes.

6°. The plane of vibration by the plane of the normals and vibrations.

132. We have now shown that when any arbitrary displacement is made in any homogeneous medium, a disturbance is propagated in all directions from the origin, and that it is materially affected and controlled by the elastic forces developed. In accepting the conclusions which result, the limitations which have been primarily established must be kept in mind, to avoid the danger of accepting these results other than as exceptional and governed by the admitted hypotheses and by the accuracy of the mathematical processes employed. Observation and experiment are essential to ascertain to what extent the corresponding physical phenomena conform to these deductions. They are to be used, when at variance, to modify the hypotheses, and ultimately through this modification to approach nearer and nearer the true theories of the physical science.

133. The fundamental hypotheses upon which the foregoing discussion is in part based are as follows:

1°. The admission of such a constitution of the medium that while it is variable around any molecule, it is similarly variable around all the molecules. The propagation of the disturbance without change of direction of the vibrations, when the latter are excited along the singular directions depends on this assumption. This inequality of elasticity is unquestionably exhibited in the phenomena of crystallization.

2°. That the excursions of the displaced molecules are so small that the resultant elastic forces in any direction are proportional to the displacement. This implies that the distances separating the adjacent molecules are very great in comparison with their displacements.

3°. The principle of the coexistence and superposition of small motions, by which any vibration can be replaced by others equivalent to it which are rectilineal.

4°. The inefficacy of the longitudinal component of the elastic force in light undulations, and the fact therefore of transversal vibrations. (The grounds of this assumption are to be given subsequently.)

5°. The correlation of the total intensity of the elastic force to certain velocities, and its identity with that expressed by the equation

$$V = \sqrt{\frac{e}{d}}.$$

6°. The principle of interference, by which the motion is entirely destroyed everywhere, except upon certain surfaces, which may be regarded as the loci of first arrival.

134. The agreement of the results obtained by experiment and from observation with the deductions from the theory is almost complete, while the crucial test of prediction in several noted instances leaves but little doubt of the truth of the undulatory theory. The utility of the determination of the wave surface and of its thorough discussion is thus happily verified, by its almost complete capability of satisfactorily explaining most, if not all, of the phenomena of physical optics. While in the limited course prescribed for the Academy we are unable to undertake the complete solution, we have, in the short and elementary discussion here presented, obtained sufficient data to prosecute the study of sound and light to the extent necessary for our purposes, and in this study we will have frequent occasion to refer to the foregoing analysis.

PART II.

ACOUSTICS.

135. The investigations of physical science show that all sensation has its origin in the state of relative motion of the molecules of some medium with which the organ of sensation is in sensible contact. Each sensation has its peculiar organ, which, with its nerve system, receives and transmits molecular kinetic energy to the brain, where it is transformed into sensation. The motions of the molecules are, in general, vibrations, which are conveyed by undulations from the source of disturbance in all directions throughout the medium.

136. Acoustics is that branch of physical science which treats of *sound.* The sensation of sound usually arises from the communication of a vibratory motion of the tympanic membrane of the ear, due to the slight and rapid changes of the air pressure upon its exterior surface, the vibratory motion of the air being caused by the vibration of other bodies.

137. The ear consists essentially of two parts, one being in communication with the external atmosphere, the other with the brain.

The first consists of an irregularly formed tube, beginning at the orifice of the external ear and ending at the pharynx. Nearly midway, the tympanic membrane, or drum-skin, of the ear crosses this tube obliquely, separating the external portion, called the *meatus,* from the part immediately within, called the *tympanum.* That portion of the tube leading from the tympanum to the pharynx, or cavity behind the tonsils, is called the *Eustachian* tube. The orifice of this tube at the tympanum is generally closed; but the act of swallowing opens it, whereupon the air on both sides of the tympanic membrane becomes uniform in density. These three portions of the first part of the ear generally, however, contain air differing in density. In the meatus the air responds to all changes, however slight and rapid, taking place in the external atmosphere; while the air in the tympanum and Eustachian tube is not so affected, unless communication with the external atmosphere be made as above described.

138. The other part, sometimes called the internal ear, is surrounded by bone, except in two places, called the *round* and *oval* windows. The cavity thus formed is called the *bony labyrinth.* The windows are closed by membranes which separate the tympanum on the one side from the fluid contained in the labyrinth on the other. Connecting the tympanic membrane with the oval window is a series of small bones, whose function appears to be to transfer the vibrations from the former to the latter. The labyrinth is filled with liquid, having suspended in it many membraneous bags, also filled with liquid. Upon the surface of these bags are spread the terminal fibres of the auditory nerves, which, by special arrangements, are enabled to take up the energy communicated to the liquid in the labyrinth. The membrane of the round window readily yields to the pressure of the liquid, moving out and in as the oval window is moved in and out by the transfer of motion through the bones of the ear.

Thus the energy communicated to the air in the external ear is conveyed from the tympanic membrane, through the series of small bones in the tympanum, to the membrane of the oval window, thence to the liquid of the labyrinth, and finally to the auditory nerves. How this energy is transformed into sensation is unknown.

139. To represent, graphically, the variations of air pressure, we will make use of the *curve of pressure,* in which the abscissas correspond to the *times* and the ordinates to the *excess* of the pressure above its mean or average value. The pressure of the air, at any point, is assumed to be measured by the pressure of air of the same density and temperature upon a unit of area. Then take

$$y = f(t) \tag{195}$$

to represent any curve of pressure as

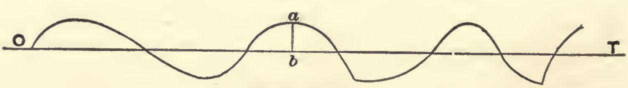


Figure 20.

in which $y = 0$ represents the standard or mean pressure, and

$y = \pm p$, a pressure above or below the standard pressure. Whenever the pressures are strictly proportional to the corresponding densities, as by the law of Mariotte, the same curve may also represent the curve of density. If we now assume that a curve similar to the above represents the slight and rapid changes of pressure of the air in contact with the tympanic membrane while the sensation of a particular sound exists, we see that these changes do not in general affect the average pressure of the air, for the areas above and below the axis of the curve are equal. A curve is said to be *periodic* when it consists of equal and like parts continuously repeated. The *wave length* of a periodic curve is the projection upon the axis of the smallest repeated portion.

140. The ear clearly distinguishes between a musical sound and a noise. The former is a uniform and sustained sensation, unaccompanied by any marked alteration, save that of intensity; while the latter is more or less varied and ununiform. When a sonorous body is sounding, the most ordinary examination is sufficient to show that it is in a state of vibration. The vibrations or oscillations of its parts set in corresponding motion the adjacent air-particles, which in turn transmit similar motions to the next following particles, and so on. The air, then, is ever passing through alternate states of condensation and rarefaction. When these vibrations are regular, periodic, and sufficiently rapid, the resulting sound is uniform in character and is called a *musical tone.* If the resulting sound arises from vibrations which are non-periodic, it is called a *noise.* Ordinary observation shows that few, if any, noises are perfectly unmusical; and few, if any, sounds are absolutely unmixed with noise.

141. ***Propagation of a Disturbance in an Indefinite Cylinder.*** Let us suppose the indefinite cylinder MN filled with air, and at the origin a piston p, capable of rapid to-and-fro motion. In the first place, let the piston be moved a distance ds from p to p', in the time dt. If the air were incompressible, it would be moved bodily over the distance ds. But

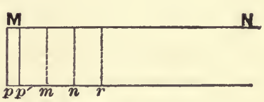


Figure 21.

being compressible, the air yields to the motion of the piston, and at the end of the time dt the compression will have reached a position m, so that the stratum of air, being condensed from pm to $p'm$, will exert an elastic force in excess of that due to its normal state. Call this excess δ. The increased elasticity of $p'm$ will cause it to expand in the only direction possible, towards the next stratum mn, which in turn becomes compressed. This second stratum reacts in both directions; on the side towards mp' it brings the molecules of mp' to rest, their acquired velocity having a tendency to cause them to pass beyond their positions of equilibrium; and on the side nr it compresses the next stratum, increasing its elasticity ultimately by δ. In this manner the compression is transmitted from stratum to stratum, throughout the whole length of the cylinder.

142. Let V be the velocity of propagation of the condensation, v the velocity of the piston; then we have

$$pp' = ds = vdt, \qquad pm = ds' = Vdt,$$

$$pm - pp' = p'm = (V - v)\,dt.$$

Supposing Mariotte's law applicable, and P to represent the normal pressure, we have

$$P : P + \delta :: p'm : pm :: (V - v)\,dt : Vdt;$$

or

$$\delta = P\frac{v}{V - v}. \tag{196}$$

Let p' now return to its primitive position p in the next successive dt. The first layer of the stratum will be dilated, occupying the new space $p'p$, and its pressure P will become $P - \delta$. The elastic force of the next layer P will become, by its expansion to the left, $P - \frac{\delta}{2}$, increasing that of the first also to $P - \frac{\delta}{2}$. But the velocity acquired by the molecules of the second layer will cause them to pass beyond their positions of equilibrium, so that its elastic force will diminish until it becomes $P - \delta$, at the instant the elastic force of the first layer, continually increasing, becomes P, its normal value. The third layer will, in turn, act on the second as the second has acted on the first, so that the dilatation corresponding to δ will travel the distance pm in the time dt, during which the piston is retracing its path $p'p$. The magnitude of δ will evidently depend on the value pp' and the time dt. If dt be constant and δ be varied, the condensations will vary with δ. The analysis shows that the compressions and dilatations are propagated with equal velocities, and that these velocities are independent of the degree of condensation or of rarefaction, when the medium is the same and the amplitude is very small.

143. Let the prong of a tuning-fork $p \dots p'$ (Fig. 22) be displaced a very small but finite distance from its neutral position a. By its elasticity it will vibrate with equal displacements on each side of its position of equilibrium. Its velocity increases from zero at p to a maximum at a, and decreases in an exactly reverse manner to zero from a to p'. Let the duration of its motion from p to p' be divided into equal parts, each represented by dt, the epoch corresponding to the position p. From p the prong describes unequal but increasing distances during the successive dt's to the position a, and unequal but decreasing distances from a to p'. Each corresponding compression can be found from Eq. (196) by the substitution of the proper value of v, and these compressions or condensations will be propagated with a constant velocity V. While the prong is returning from p' to p, the rarefactions will increase from p' to a, and decrease from a to p, and their values may be determined from the same equation. The condensations will be symmetrically distributed with reference to the maximum condensation, neglecting the very small amplitude vp'. Likewise the rarefactions will be symmetrically distributed with respect to the maximum rarefaction.

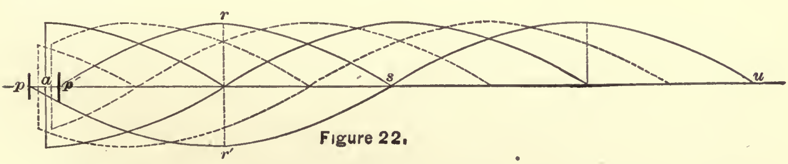


Figure 22.

144. The positive ordinates of the curve $p'rs$ represent the successive condensations, s being the position of the layer reached by the first condensation when the prong has arrived at p'; and the negative ordinates $pr's$ will represent the successive rarefactions when the first condensation has reached the position u, and the prong has returned to its primitive position p. The ordinates of the other curves represent either condensations or rarefactions, as indicated in the figure corresponding to the particular state and position of the prong.

145. In the figure, *pu*, the length of the wave is the distance traveled by the disturbance while the prong is making a complete vibration, and hence we have, n being the vibrational number and τ the periodic time,

$$\lambda = \frac{V}{n} = V\tau. \tag{197}$$

146. The mean velocity of the air molecules is evidently the same as that of the vibrating prong, and therefore this will vary with the vibrating body. In the example given, the mean velocity of the molecules is 2 mm. $\times 256 = 0.512$ m. The actual velocity of the air molecules continually varies, and at any time is proportional to the ordinates of the curve a quarter of a wave length in advance of the molecules considered. When the vibrating body has simple harmonic motion, the molecular velocity is given by

$$v = a \cos 2\pi \frac{t}{\tau}. \tag{198}$$

147. The value of a, the amplitude of the vibration, diminishes (Art. 72) according to the law of the inverse distance from the centre of disturbance; and for each value of a taken as constant within the wave length we have, by the above equation, sensibly exact values for the molecular velocity at any time.

148. When the vibrations of the body are sufficiently frequent during the unit of time, and of sufficient amplitude, the sensation of sound arises in the ear, which, however, we unconsciously refer to the vibrating body. A sonorous wave comprises the series of condensations and rarefactions arising from one complete vibration of the sounding body.

149. The sum of all the condensations in the condensed portion of the wave is represented by the area of the curve $p'rs$, and if it be divided by the duration of half the vibration, the mean condensation will result. Thus, take the amplitude of the oscillation of the tuning-fork, making 128 vibrations per second to be 1 mm., and the velocity of propagation to be 340 m.; then, from Eq. (196), we will have

$$\delta = P\frac{v}{V-v} = P\frac{\frac{1}{\frac{1}{256}}}{340000 - \frac{1}{\frac{1}{256}}} = P\frac{256}{340000-256} = P\frac{1}{1327}. \tag{199}$$

Hence, the change of density in the air, measured by the barometric height due to the mean condensation, is not greater than that due to 0.0226 inches of mercury, when a sound corresponding to 128 vibrations per second, and caused by the fork under the supposed conditions, is passing.

150. From the preceding discussion we see that we can neglect, in general, the absolute displacements of the air molecules, and consider the change in pressure and density as being alone propagated. Therefore, a file of elastic balls transmitting motion practically illustrates the state or condition of a series of air molecules during the propagation of a sonorous wave. An excellent illustration is also given by means of a chain cord. If it be attached at one of its extremities to a fixed point, and be held stretched at the other, the successive rings or spirals will assume positions of stable equilibrium with respect to each other, determined by the tension. These rings, for the purpose of illustration, may be taken to represent the contiguous air strata or particles in an indefinite tube, or upon any line along which sound is supposed to be propagated. If any ring be plucked, it will, when released, oscillate about its place of rest while the disturbance is being propagated in both directions to the points of support. Upon reaching these points the disturbance will be divided, a part proceeding in the new medium, and the remainder, being reflected, will retrace its path, to be again subdivided at the other end. This will continue until the whole energy of the original disturbance has been dissipated. By increasing the tension the disturbance will be more quickly propagated, and conversely. Now suppose, from the point of plucking, lines be drawn in all directions, and the same phenomena occur on these, then the behavior of each ring and the progressive motion of the disturbance illustrates what takes place in air during the passage of a sound wave along every right line drawn from the origin of the sounding body. In an isotropic and homogeneous medium, the disturbance moves with constant velocity, and the volume whose surface bounds the disturbed particles at any instant is a sphere whose radius is Vt.

151. The general properties of any sound are *intensity, pitch,* and *quality*.

Intensity is that property by which we distinguish the relative loudness of two tones of the same pitch and quality. We can also, in general, determine which of two tones of different pitch and quality has the greater intensity. The air particles have but small displacements from their positions of relative rest, when the displacement is caused by the passage of a sound wave. The forces which urge them back to their positions of rest are assumed to vary directly with the degree of displacement. In Analytical Mechanics, it is shown that the periodic time of the air particle depends only upon its mass and the intensity of the force of restitution; and therefore, in the same medium, with given pressure, density, and temperature, for the same exciting cause, the periodic time will be constant, but the mean velocity of the air particle will vary with the size of the orbit. The kinetic energy in the moving particle, varying as the square of the velocity, will therefore, for the same exciting cause and the same medium, under the same circumstances of pressure, density, and temperature, vary directly as the square of the maximum displacement. By the law of the decay of energy, the intensity of the sound will therefore vary inversely as the square of the distance from the origin of the exciting cause. (Art. 72.)

152. *Pitch* is that property by which we distinguish the position of two tones in the musical scale, and thereby recognize which is the more acute and which the more grave. The pitch depends upon the frequency of the vibration; the greater the number of vibrations produced by a sounding body in a given time, the more acute will be the resulting sound. The siren is an instrument used to illustrate this fact. It consists essentially of a disk pierced with a number of equidistant holes, through which air is forced when it is put in rapid rotation. As the rotation increases, the sound gradually rises in pitch, and as it diminishes the pitch falls correspondingly. If a coin with a milled edge, or a cogged wheel, be put in rotation, and a card be held against it, the same changes in pitch will be observed. In these cases the single puff, or stroke of the card against the coin, or wheel, is essentially a noise, and when these strokes are multiplied sufficiently in a given time, the resulting effect is a note of definite pitch. So that a clearer distinction than that heretofore given should be made between a noise and a musical tone. To this distinction we will again refer.

153. The *quality* of a musical tone is that property by which we can distinguish whether two sounds of the same pitch, of either equal or unequal intensities, arise from the same or different sonorous bodies. This property enables us, within certain limits, to distinguish voices and the various sounds peculiar to different musical instruments. We have as yet only exacted that a musical sound shall be periodic and regular; that is, that during any vibration the successive states of motion of the particle shall recur in the same order as in each of the previous vibrations. But it is evident that we may have an infinite variety of periodic motion, and it will be shown that the quality of the sound will vary with each variation of the periodic motion, the wave length remaining constant.

154. Every one has experienced the fact that more than one sound can be heard at once. Our attention can be, for the moment, fixed upon any one of the many sounds that are constantly occurring, and at the same time we may be conscious of the existence of the others. Therefore, the meeting of sound waves in the external ear does not, in general, result in mutual destruction, or in essential modification; while, at the same time, we must acknowledge that the air in contact with the tympanic membrane, at any given instant, can possess but one determinate pressure and density. The changes in pressure and density due to many exciting causes must, then, result from the superposition and coexistence of those arising from each separate cause, and, in general, without destruction or modification. We have here the application of the principle enunciated in Art. 204, Mechanics.

The more general statement of the law of the composition of displacements would be that demonstrated in the principle of the parallelogram of forces, but when the displacements are infinitely small, we can take, rigorously, the resultant displacement to be the algebraic sum of the component displacements. The diameter of the meatus at the tympanic membrane does not exceed 0.25 inch, and therefore, for sounds whose sources are at ordinary distances, the wave fronts at the position of the tympanic membrane coincide sensibly with their tangent planes, and the changes of density and pressure may be compounded by the law of small motions, without appreciable error.

155. Let the broken line, in the following diagram, represent the changes of pressure upon the tympanic membrane while a continuous noise, in which the ear recognizes no definite pitch, is sounding for a small part of a second, and let the dotted line represent another noise of the same duration.

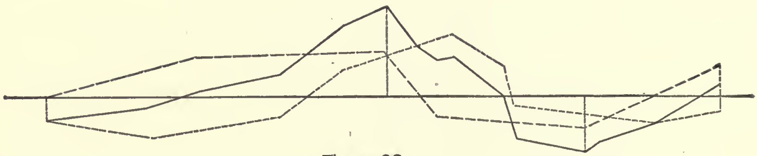


Figure 23.

Then, if both noises sound together, the resultant variation of pressure will be represented by the full line obtained by joining the extremities of the ordinates found by taking the algebraic sum of the ordinates of the separate curves.

These two noises do not, in general, unite into one, but are heard distinctly and simultaneously, except in the case where the two sounds are nearly alike, and the two curves nearly similar. Again, there is nothing in the resultant curve to suggest to the eye the nature of the two component curves. Hence, the ear possesses the property of separation; while the eye, according to this method of combination and representation, does not.

156. Let the component curves be periodic, two periods of $O'X'$ being equal to three of $O''X''$.

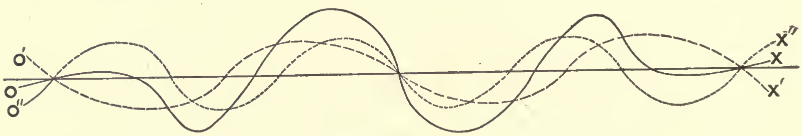


Figure 24.

The resultant curve OX will be a periodic curve, whose repeated portions are represented above. An examination of this curve by the eye gives no clue to its components, and we may resolve it into an indefinite number of pairs of components, but one of which would represent the two notes which sounding together will give us the resulting effect upon the ear. But if the ear resolves the composite note represented by OX, it must resolve, in like manner, $O'X'$ and $O''X''$. Observation confirms this deduction.

157. The only note the ear is incapable of resolving is that of the simple musical tone, and this incapability arises from the fact that such a tone is in reality perfectly simple, and not compound. The tones which are ordinarily called simple, are, in reality, compounded of a series of simple tones theoretically unlimited in number. Very few of them have sufficient intensity to be heard; but these few form a combined note which is always the same under the same circumstances, and we habitually associate them together, and perceive them as a single note of a special character. But it is possible, with certain appliances, to partially analyze the composite note by an attentive study of the separate constituents.

Whenever two sounding bodies give notes whose tones form *consonant* combinations with each other, the difficulty of analysis is increased; when the combinations are *dissonant*, the analysis is less difficult.

158. A noise may therefore be defined to be a combination of musical tones, too near in pitch to be separately distinguished by the unassisted ear, or to be a combination of noises, each of which is made up of sounds so near each other in pitch as to be undistinguishable; the separate noises may be near or far apart in pitch. It is so complex, that its analysis is beyond the power of the unassisted ear. A simple musical tone, on the contrary, is incapable of resolution by reason of its absolute simplicity. Hence, strictly speaking, only simple tones have pitch. A simple musical tone will have a single determinate pitch. The pitch of a musical note must then be taken to mean the pitch of the *gravest* simple tone in its combination. If the higher simple tones be successively stopped out, the pitch, as defined, will remain unaltered, but the quality of the note will undergo variations until the single musical simple tone corresponding to the gravest tone is reached, beyond which no further modification can take place.

159. We will hereafter assume, as the fundamental simple tone, that component of any note which corresponds to the regular periodic curve of the given pitch. This distinction is important; for it is evident that there may be many periodic curves of the same pitch, and each may correspond to musical notes differing in quality.

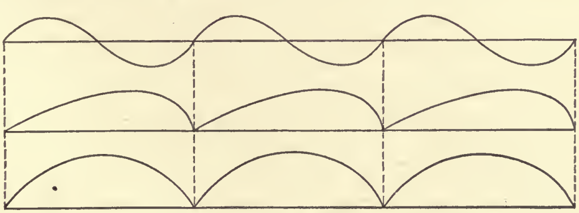


Figure 25.

The preceding curves (Fig. 25) represent notes of the same pitch, but of different quality. Helmholtz has shown that while every different quality of tone requires a different form of vibration, the converse is not necessarily true; *i. e.*, that different forms of vibration may not correspond to the same quality.

160. We have seen, page 45, that any physical condition, such as density, pressure, velocity, etc., which is measurable in magnitude or intensity, and which varies periodically with the time, may, by Eq. (195), be expressed as a function of the time. Hence, every periodic disturbance of the air, and particularly such disturbances as excite the sensation of a musical tone, can be resolved into its harmonic vibrations.

A single simple tone being represented by the simple harmonic curve

$$y' = a' \sin\left(\frac{2\pi x}{\lambda} + \alpha'\right), \qquad (200)$$

and another of half wave length by

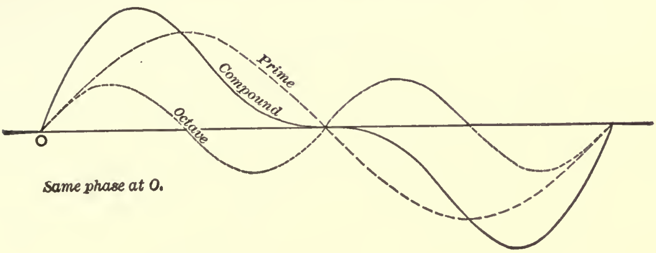
$$y'' = a'' \sin\left(\frac{2\pi x}{\frac{\lambda}{2}} + \alpha''\right) \qquad (201)$$

the resultant curve will be represented by

$$y = a' \sin\left(\frac{2\pi x}{\lambda} + \alpha'\right) + a'' \sin\left(\frac{4\pi x}{\lambda} + \alpha''\right), \qquad (202)$$

which has the same wave length, but a different amplitude and phase. This change in the amplitude and phase may be varied at pleasure, by conceiving the second curve to be shifted along the axis any distance from zero to λ, and again to pass through all values of the amplitude between any two limits. The resultant curve, in all cases, will, however, be a periodic curve of constant wave length.

161. Considering the simple musical tones which they represent then to be sounded together, with the same modifications, it has been found that the ear can distinguish the components when the attention is cultivated and directed to this effect. With a variation in phase only, the effect on the ear is constant and invariable, and hence we see that many different resultant curves may represent



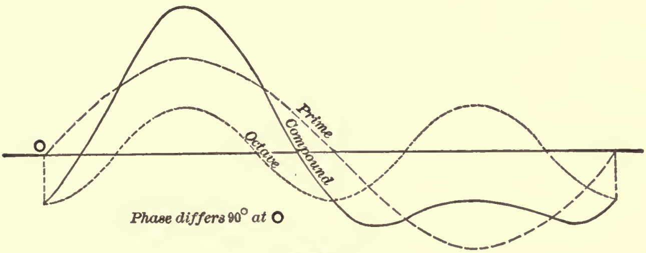


Figure 26.

essentially the same sensation. Thus, the two curves above, represent the same compound tone made up of the two simple tones, although the forms of the curves are quite different. The resultant tones are the same, both in quality and in pitch, but differ in intensity. By combining in the same way other simple tones of one-third, one-fourth the wave length, and so on, the quality will be changed, without affecting the pitch, as can be seen from the graphical construction, and heard by audible experience. In all these cases the untrained ear, by the aid of certain appliances, can always analyze the resultant sound into its component simple tones, and when trained, often without this assistance. When but one simple vibration of sufficient frequency and intensity to produce sensation alone exists, no such analysis takes place.

162. The investigations of Helmholtz have shown that the ear possesses the property of analysis of a single musical tone into its simple musical tones, each of which is distinctive in character, but which blend harmoniously into the single tone when sounded together. The wave lengths of these components are aliquot parts of the wave length of the fundamental, and the simple tones are called the *upper partials* of the fundamental or prime tone. Hence, from Art. 64 and these facts, we conclude that, when several sounding bodies simultaneously excite different sounds, the variations of air density and the resultant displacements and velocities of the air particles in contact with the tympanic membrane are each equal to the algebraic sum of the corresponding changes of density, the displacements and the velocities which each system of waves would have separately produced, had it acted alone.

163. This analysis by the ear clearly shows, then, that the separate effects of the simple vibrations are, in general, neither modified nor destroyed, but actually exist, and it remains to be proved that such is really the case, independent of the peculiar sensation which is the result of their action upon the ear. Since Fourier's Theorem mathematically demonstrates that any form of vibration, no matter how varied its shape, can be expressed as the sum of a series of simple vibrations, its analysis into these simple vibrations is independent of the capacity of the eye to perceive by examining its representative curve whether it contains the simple harmonic curves or not, and if it does, what they are. All that the curve indicates is that the more regular its form, the greater the effect of its deeper or graver tones in comparison with its upper partials. Before proceeding to show that these component vibrations actually exist together, and that each can affect the ear or other sensitive vibrating body, let us now establish clearly the definitions pertaining to the subject.

164. A *simple* or *pendular* vibration is that which corresponds to the complete oscillation of a simple pendulum, and is graphically represented by the simple harmonic curve.

A *simple musical tone* is that effect produced upon the ear when a sonorous body is executing simple vibrations only, of sufficient frequency and amplitude to be heard. According to this definition, simple tones do not in reality exist ; but in the vibrations of such bodies as tuning-forks, the component vibrations which simultaneously exist with that of the gravest period, are generally nonperiodic with it, and so deficient in intensity that their influence is negligible, and we may regard such bodies as producing simple vibrations alone without sensible error.

165. A *single* musical tone may be either simple or compound. When compound, it is made up of its fundamental simple tone, together with its upper partial simple tones, each of which has a frequency of either twice, three times, or so on, that of its fundamental. It is due to the vibration of a single sonorous body which, during its motion, vibrates as a whole, and divides also into parts which vibrate twice, three times, and so on, as rapidly as the whole. One or more of these upper partials may be wanting during the vibration; when this occurs, the quality of the single musical tone is correspondingly affected.

A *composite* musical tone is composed of two or more single musical tones.

166. ***Musical Intervals.*** The extreme range of the human ear lies between 20 and 40000 simple vibrations per second. The corresponding wave lengths are obtained by dividing the velocity of sound by these numbers, and are approximately 54.6 feet and 0.0273 feet respectively, assuming the velocity of sound to be 1092 feet at 0° C. The ordinary sounds heard by the ear have a much less range; their vibrational numbers lie between 40 and 4000, corresponding to wave lengths of about 27.3 feet and 0.273 feet, respectively. When a stretched wire is put into vibration, and the tension continuously undergoes variation, the pitch of the sound passes by continuity from lower to higher, or the reverse, and we therefore experience the sensation of a musical interval between any two limiting tones. We may, then, define a musical interval by the ratio of the vibrational numbers of the two limiting tones. Thus, if the two tones correspond to the vibrational numbers 256 and 384, the name of the interval is the *fifth*, and it is expressed by the fraction $\frac{3}{2}$. Considering the simpler ratios that lie between two tones whose vibrational numbers are as 1 : 2, we obtain the following musical intervals:

Consonant.			*Dissonant.*		
Unison, . . .	1 : 1	$= \frac{1}{1}$	Major second, . . .	9 : 8	$= \frac{9}{8}$
Minor third, . .	6 : 5	$= \frac{6}{5}$	Minor second, . . .	10 : 9	$= \frac{10}{9}$
Major third, . .	5 : 4	$= \frac{5}{4}$	One-half major tone,	16 : 15	$= \frac{16}{15}$
Fourth, ...	4 : 3	$= \frac{4}{3}$	One-half minor tone,	25 : 24	$= \frac{25}{24}$
Fifth, ...	3 : 2	$= \frac{3}{2}$	Comma, ...	81 : 80	$= \frac{81}{80}$
Major sixth, . .	5 : 3	$= \frac{5}{3}$			
Octave, ...	2 : 1	$= \frac{2}{1}$			

The first are called *consonants*, because the effect is pleasing to the ear when the tones of either of these intervals are sounded together. All other intervals within range of the octave are called *dissonants.*

167. The measure of the musical interval represented by the ratio $\frac{p}{q}$ is the $\log\frac{p}{q}$. This arises from the fact that if we consider any three tones whose vibrational numbers are p, q, and r, the musical interval between p and r must be equal to the sum of the two intervals between p and q, and q and r. If the ratios of the vibrational numbers were taken to measure the intervals, we would have, for the same interval, the expressions

$$\frac{r}{p} \quad \text{and} \quad \frac{q}{p}+\frac{r}{q}.$$

which are not equal to each other. But since

$$\frac{r}{p}=\frac{q}{p}\times\frac{r}{q}, \tag{203}$$

we have

$$\log\frac{r}{p}=\log\frac{q}{p}+\log\frac{r}{q}, \tag{204}$$

and we may therefore take the logarithm of the ratio of the vibrational numbers as the measure of the musical interval. The *name* of any interval, then, is the ratio of the vibrational numbers, and its *measure* is the logarithm of that ratio. The logarithms are usually taken in the common system.

168. ***Musical Scales.*** A series of tones at finite intervals is called a musical scale. If the vibrational numbers are in the proportion of the natural numbers, the musical scale is called the *harmonic* scale. When two tones whose interval is that of an octave are sounded together, we are conscious of a certain *sameness* of sensation, which is absent in all other intervals except multiples of the octave. We may then assume this interval as a natural unit, since it gives a periodic character to the scale. Whatever properties are found with regard to the tones in any octave, occur in the other octaves of a higher or lower pitch. The vibrational numbers of the tones of the harmonic scale, starting with a fundamental tone whose vibrational number is 128, will be as follows:

128 : 256 : 384 : 512 : 640 : 768 : 896 : 1024 : 1152 : 1280 : etc.

169. Examining these numbers, we see that each interval in any octave is divided, in the succeeding octave, into two intervals which can be obtained from the equation

$$\frac{n+1}{n} = \frac{2n+1}{2n} \times \frac{2n+2}{2n+1}, \qquad (205)$$

n being the natural number which marks the position of the first tone of the lower interval in the harmonic scale. Thus we see that the interval 128 : 256, or the octave, is divided in the next octave into two intervals represented by $\frac{2n+1}{2n} = \frac{3}{2} = \frac{384}{256}$ and $\frac{2n+2}{2n+1} = \frac{4}{3} = \frac{512}{384}$. The first interval, 256 : 384, in the second octave is divided into the two intervals corresponding to $\frac{2n+1}{2n} = \frac{5}{4}$ and $\frac{2n+2}{2n+1} = \frac{6}{5}$ in the third octave; the second interval, 384 : 512, in the same octave, is in like manner divided into $\frac{2n+1}{2n} = \frac{7}{6}$ and $\frac{2n+2}{2n+1} = \frac{8}{7}$ in the third. The first interval, 512 : 640, in the third octave, is subdivided in the fourth octave into $\frac{9}{8}$ and $\frac{10}{9}$, and so on. Arranging all the intervals, with their corresponding subdivisions in the next higher octave, we have

1st octave, 128 : 256, interval $\frac{2}{1}$, subdivided in 2d octave into

$$256 : 364 = \frac{3}{2} \quad \text{and} \quad 384 : 512 = \frac{4}{3};$$

2d octave, 256 : 384, interval $\frac{3}{2}$, subdivided in 3d octave into

$$512 : 640 = \frac{5}{4} \quad \text{and} \quad 640 : 768 = \frac{6}{5};$$

2d octave, 384 : 512, interval $\frac{4}{3}$, subdivided in 3d octave into

$$768 : 896 = \frac{7}{6} \quad \text{and} \quad 896 : 1024 = \frac{8}{7};$$

3d octave, 512 : 640, interval $\frac{5}{4}$, subdivided in 4th octave into

$$1024 : 1152 = \frac{9}{8} \quad \text{and} \quad 1152 : 1280 = \frac{10}{9};$$

3d octave, 640 : 768, interval $\frac{6}{5}$, subdivided in 4th octave into

$$1280 : 1408 = \frac{11}{10} \quad \text{and} \quad 1408 : 1536 = \frac{12}{11}.$$

Thus every interval in the harmonic scale is divisible into two other intervals, whose ratios are those of consecutive numbers in the next higher octave.

170. ***Perfect Accords.*** A *perfect accord* is a series of three tones, called a *chord*, which, sounded simultaneously, give a particularly pleasing sensation to the ear. The *perfect major accord* consists of the three tones called the *tonic*, the *middle*, and the *dominant*, whose intervals are a major third and a fifth, or $\frac{5}{4}$ and $\frac{3}{2}$.

The *perfect minor accord* is composed of a minor third, $\frac{6}{5}$, and a fifth, $\frac{3}{2}$.

171. ***The Diatonic Scale.*** The tones of this scale are usually designated by letters or symbols, as follows :

C : D : E : F : G : A : B : *c* : *d* : etc.

ut or *do* : *re* : *mi* : *fa* : *sol* : *la* : *si* : *do* : *re* : etc.

Forming the perfect major accord on C as a tonic, we will have

$$\text{C} : \text{E} : \text{G},$$

$$1 : \frac{5}{4} : \frac{3}{2}.$$

Forming similar chords with C and G, by making C a dominant and G a tonic, we will have

$$F_1 : A_1 : C, \qquad G : B : d,$$

$$\frac{2}{3} : \frac{5}{6} : 1; \qquad \frac{3}{2} : \frac{15}{8} : \frac{9}{4}.$$

Arranging these three chords in order of their pitch, we find

$$F_1 : A_1 : C : E : G : B : d,$$

$$\frac{2}{3} : \frac{5}{6} : 1 : \frac{5}{4} : \frac{3}{2} : \frac{15}{8} : \frac{9}{4},$$

which is a musical scale of seven notes, rising one above another by alternate major and minor thirds.

Replacing in this scale F_1, A_1, by their higher octaves, and d by its lower octave, which is permissible, and arranging in order, we have

$$C : D : E : F : G : A : B : c,$$

$$1 : \frac{9}{8} : \frac{5}{4} : \frac{4}{3} : \frac{3}{2} : \frac{5}{3} : \frac{15}{8} : 2,$$

which is known as the diatonic scale. The names of the intervals heretofore used are now seen to come from the position of the notes in this scale with reference to the tonic; thus, the interval $\frac{9}{8}$ is a major second, the interval $\frac{5}{4}$ a major third, $\frac{4}{3}$ a fourth, $\frac{3}{2}$ a fifth, and so on. The first tone in the scale is called the *tonic*, the fifth the *dominant*, and the fourth the *subdominant*. Taking the vibrational number of the tonic C to be 24, we have the corresponding vibrational numbers of the diatonic scale,

$$C : D : E : F : G : A : B : c,$$

$$24 : 27 : 30 : 32 : 36 : 40 : 45 : 48.$$

172. The vibrational numbers of the other octaves are obtained from these by constantly doubling or halving them, according as we ascend or descend, the letters being properly accented to indicate in which octave the series is taken. Theoretically, the tones of the diatonic scale above belong to the harmonic scale, whose fundamental tone has *one* vibration per second. This fundamental tone is *five* octaves below the subdominant; for $\frac{32}{1} = \left(\frac{2}{1}\right)^5$. We will hereafter take the octave whose tonic corresponds to 256 vibrations for that of comparison, because Scheibler's tonometer, which we use in illustration in the lectures on this subject, is based on that tonic.

173. The relation of the successive tones of the harmonic scale to any tone assumed as a fundamental is as follows; taking as the prime that whose vibrational number is 256, we have

Prime or fundamental,	256	vibrations,	or *c*;
1° Harmonic,	512	"	" *c'*, octave;
2° "	768	"	" *g'*, fifth in 1st octave;
3° "	1024	"	" *c''*, second octave;
4° "	1280	"	" *e''*, maj. third in 2d oct.;
5° "	1536	"	" *g''*, fifth of 2d octave;
6° "	1792	"	" *a''*+, lying between 6th and 7th of 2d oct.;
7° "	2048	"	" *c'''*, third octave;

and so on. These harmonics are called *overtones* or *upper partials*, and, as seen above, bear a close relationship to the prime. When the prime is sounded and the upper partials exist at the same time, the resulting tone will have a determinate quality. And if the partials be successively stopped out, the quality will undergo a change, until we reach the simple tone due to the prime alone. The successive curves which represent these tones graphically will approximate gradually to that of the harmonic curve of the wave length of the prime, which it ultimately reaches when all of the partials are wanting. The wave lengths of the above curves are each equal to that of the prime.

174. It can be experimentally shown that a stretched cord, when plucked from its position of rest, will give a compound tone, which is made up of its fundamental united to some of its overtones. The educated ear can readily distinguish the existence of these simple tones, which, sounding together, determine the quality of the compound tone. But to demonstrate to the untrained ear the existence of these partial tones, it is necessary to make use of certain appliances called *resonators,* whose action depends on the principle of *sympathetic resonance.* These consist of metal or other hollow bodies, generally spherical in form, closed except at two places; one of the openings is to permit the mass of air within to be affected by the vibration of the air without, and the other to permit the air within to be brought into near contact with that in the aperture of the ear.

175. ***Sympathetic Resonance.*** If a body capable of taking up an oscillatory motion of definite period be subjected to a series of periodic impulses, whose period is the same as that of the body considered, the aggregate effect will in time become sensible, however weak the impulses may be. But if the period of the impulses be even slightly different from that of the body, the resultant effect will, in general, never become appreciable; for, while the kinetic energy is increased by the elementary quantities of work due to the impulses applied, soon the succeeding impulses will be delivered in a direction contrary to the motion of the body, and the kinetic energy will be correspondingly diminished. The maximum energy can then never exceed a small definite quantity, and in reaching this state the body will pass through alternations of rest and motion. To determine the effect of any periodic impulse upon a body capable of being put into vibration, we have the following rule, due to Helmholtz: Resolve the periodic motion of the impulse into its component simple pendular vibrations; if the periodic time of any one of these vibrations is equal to the periodic time of the body acted upon, sensible vibration will result, and not otherwise.

176. Now consider the mass of air within the tube AB, while a simple vibratory motion, due to a simple tone, occurs in the external air. Let V be the velocity of wave propagation in the air under consideration, and n the vibrational number of the body. Then, during the first semi-vibration, the molecules at B describe half their orbits while undergoing condensation, which is transmitted through the intervening molecules to A and back to B, provided

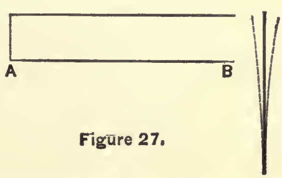


Figure 27.

$$BA + AB = \frac{V}{2n}.$$

During the second semi-vibration, the rarefaction at B will be transmitted in the same manner, and the orbits at B will be completed. Should BA be either > or < $\frac{V}{4n}$, the second impulse would reach B after or before its molecular orbits had been completed. Under these circumstances, succeeding impulses would in a short time reduce the displacements of the molecules to zero, and never permit them to attain an appreciable value, and therefore the vibration of the air column would not give a sound of appreciable intensity. But if, on the contrary, the impulses were of the same periodicity as the air molecules, each successive impulse would add to the first displacement, and this addition would continue until the work of the resistances developed was exactly equal to the increment of energy caused by each impulse. The displacements of the molecules would then have attained their maximum value, and the resulting sound a fixed intensity.

177. Each confined mass of air has a particular periodicity, and each of the resonators of Helmholtz is carefully contrived to respond to a given periodicity of vibratory motion. If, then, by the rule above given, any composite sound exist, and one of these resonators be applied to the ear, the resonant effect will indicate whether the simple tone corresponding to the resonator is present or absent in the composite sound. This and analogous experiments show that sympathetic vibration is not due to any property peculiar to the ear, but that it is a mechanical effect separate and distinct from the sense of audition.

178. The energy of motion depending upon the mass and velocity, we see clearly that of two sounding bodies, vibrating with the same amplitude, the smaller mass will more quickly give up its energy to the surrounding air and sooner cease sounding. Tuning-forks being generally made of steel, will, when put into rather strong vibration, continue sounding for a reasonable length of time. When mounted upon their resonant boxes, the latter containing a mass of air capable of vibrating in unison with it, they affect larger masses of air than when not so mounted, and come more quickly to rest; but the sound will have greater intensity, and can the more readily be used to study the phenomena of sympathetic resonance. If such a tuning-fork be in the vicinity of a vibrating sounding body whose sound contains the tone of the fork, the latter will in time indicate the fact by coming into sympathetic vibration. The analysis, then, of any composite note can be practically made by means of a sufficient number of such forks, whose vibrational numbers embrace all the simple notes of the composite sound. Conversely, the synthesis of a composite note can be effected by setting in vibration all the forks, with proper amplitudes, which the analysis indicates belong to the note in question.

179. When plates, bells, strings, etc., are put into vibration, they may either vibrate as a whole, or separate into parts which vibrate two, three, four, or more times as rapidly; or both of these conditions may occur simultaneously. Each of the simple periodic vibrations has an actual existence, and corresponds to a single musical tone of definite pitch, which may be recognized as above described.

180. In listening for any simple tone in the composite note, it is important to clearly fix the attention upon the special tone whose existence is to be determined, and for this purpose the tone should be sounded alone before listening for it in the composite note. When sufficiently practiced in this manner, the ear can readily acquire the faculty of detecting them without the use of resonators.

181. By means of the *monochord,* which consists essentially of a string stretched over two bridges on a sounding-box, we can verify the simultaneous existence of the prime and upper partials, and estimate the influence of the latter in affecting the quality of the sound. The theory of vibrating strings shows that the frequency of vibration of the same string under the same tension is inversely proportional to its length. Plucking the string at its centre, the resulting tone will be that of its prime, modified by some of the upper partials, those of the latter being absent that require the middle point as a point of rest. By a movable bridge, the string can be divided into its aliquot parts, which being set in vibration, will give the upper partials in succession. Becoming thus acquainted with these simple tones, we can verify their presence or absence in each special case. For example, if the string be plucked at one-fourth its length, theory requires the presence of the first upper partial with the prime, and the fact will be made manifest by damping the string at the middle point immediately after plucking, when the octave will sing out, no longer encompassed by the prime.

182. These and the facts of sympathetic resonance show that the analysis of all resonant motion into simple pendular vibrations is real and actual, and that any other analysis is highly improbable. The analogous property of the ear is expressed by the law of G. S. Ohm, viz., *that the human ear perceives pendular vibrations alone as simple tones, and resolves all other periodic motions of the air into a series of pendular vibrations, hearing the simple tones which correspond to these simple vibrations.* We may therefore conclude, that in all cases whenever any motion of the air caused by a sounding body contains a simple vibration of the same periodicity as that of any other body, the latter will in time take up a vibratory motion which, if of sufficient intensity, will affect the ear with a simple musical tone of a definite pitch; and the mechanical effect of vibration will ensue, whether it be of sufficient amplitude to produce a sonorous effect or not.

183. ***Velocity of Sound in any Isotropic Medium.*** The air is the medium of transfer to the ear of the vibratory motion of a sounding body. Under a given temperature and density, its elastic force is constant in all directions, and it is therefore an isotropic medium. Being compressible, the motions of its molecules, during the passage of a sound wave, are to and fro along the line of wave propagation. They are then longitudinal vibrations, and Eq. (116), for the velocity of wave propagation, for waves with such vibrations in an isotropic medium, is

$$V^2 = \frac{\lambda^2}{2\pi^2} \Sigma\mu \left[\phi(r) + \psi(r) \frac{\Delta x^2}{r^2} \right] \sin^2 \frac{\pi}{\lambda} \Delta x. \qquad (206)$$

184. The wave lengths of sound in air can never be greater than 54.6 ft., nor less than 0.027 ft.; for the usual sounds the limits are 27.3 ft. and 0.273 ft., at 0° C. In the above equation, Δx is the distance separating two adjacent molecules, and without knowing its absolute value for any degree of pressure, we may say that λ, even in the minimum sound wave, is very great with respect to Δx. Therefore the arc is approximately zero, and may be substituted for $\sin \frac{\pi \Delta x}{\lambda}$ without appreciable error. We then have

$$\left.\begin{aligned} V^2 &= \frac{\lambda^2}{2\pi^2}\Sigma\mu\left[\phi(r)+\psi(r)\frac{\Delta x^2}{r^2}\right]\frac{\pi^2\Delta x^2}{\lambda^2} \\ &= \frac{1}{2}\Sigma\mu\left[\phi(r)+\psi(r)\frac{\Delta x^2}{r^2}\right]\Delta x^2. \end{aligned}\right\} \tag{207}$$

Hence, with the supposition of small displacements, etc., the velocity of wave propagation of sound in air is theoretically independent of the wave length, and all sounds, whether grave or acute, will travel, in air of constant pressure and temperature, with equal velocities.

Omitting the term containing Δx^4 as being small compared with that of which Δx^2 is a factor, and replacing $\phi(r)$ by its equal $\frac{f(r)}{r}$, we have

$$V^2 = \frac{1}{2}\Sigma\mu\frac{f(r)}{r}\Delta x^2. \tag{208}$$

185. Let E represent the modulus of longitudinal elasticity of air, P the barometric pressure, l the length of the air column without pressure, and λ the compression due to P. Then, by Eq. (1), we have

$$E = \frac{l}{\lambda}P. \tag{209}$$

Since, if the pressure P be removed, the expansion would be indefinitely great, the compression λ is sensibly equal to l, and therefore

$$E = P; \tag{210}$$

that is, the elastic force of the air is that due to the barometric pressure on the unit area.

186. In Eq. (208), $\frac{1}{2}\Sigma\mu f(r)$ is the acceleration due to the aggregate elastic forces developed in the molecules μ by the arbitrary displacement of the molecule m, and reciprocally is the elastic acceleration of m; hence we have, by multiplying by m, the intensity of the elastic force acting on m,

$$\frac{1}{2}m\,\Sigma\mu f(r);$$

$\frac{\Delta x}{r}$ is the cosine of the angle made by this force with the axis of x, which, since the medium is isotropic, is equal to unity; multiplying the elastic intensity on m by the factor $\frac{1}{\Delta x^2}$, we have the elastic intensity on the unit area, or

$$\frac{1}{2} m \, \Sigma\mu \, f(r) \frac{1}{\Delta x^2} = E; \tag{211}$$

whence,

$$\frac{1}{2} \Sigma\mu \, f(r) = \frac{E \, \Delta x^2}{m}. \tag{212}$$

Substituting in Eq. (208), we have

$$V^2 = \frac{E \, \Delta x^3}{m}; \tag{213}$$

Δx^3 is the volume of the molecule, and replacing it by its equal $\frac{m}{D}$, and extracting the square root, we will have, finally, for the velocity of wave propagation in air or any gas, subjected to the law of Mariotte,

$$V = \sqrt{\frac{E}{D}}, \tag{214}$$

or directly proportional to the square root of the ratio of the elasticity of the medium to its density.

187. This conclusion is deduced on the hypothesis of the direct ratio of the elastic force to the density, and if the law of Mariotte were true for all circumstances of pressure, temperature, and density, this theoretical velocity and the actual velocity determined by experiment would perfectly accord. But this relation is true only for a perfect gas and for constant temperature.

188. The relation of these two parameters of air, considered as a perfect gas, are given by the following formulæ :

$$p'd = pd', \tag{215}$$

$$p'd = pd'(1 + \alpha\theta), \tag{216}$$

$$p' = p\left(\frac{d'}{d}\right)^{\gamma}, \tag{217}$$

in which p and p' are respectively the old and the new pressures or the corresponding elastic forces, d and d' the old and the new densities; α the coefficient of expansion, a constant, and equal to $\frac{1}{273}$ for Centigrade scale; and θ, degrees of temperature Centigrade.

The first of these equations is the mathematical expression of Mariotte's law; the second, of that of Charles or Gay-Lussac; and the third, of that of Poisson. The gas to which these equations are applicable is supposed to be a perfect fluid, devoid of friction, and to have the pressure at each point uniform in all directions.

The temperature is supposed constant during all changes of pressure and density in Mariotte's formula, while in that of Charles the gas takes the pressure and density determined by the change of temperature. The formula of Poisson supposes the gas subjected to sudden changes of density, and that the heat developed, whether considered positively or negatively, is not conveyed by radiation or conduction to other bodies, or, in other words, that the quantity of heat in the gas is constant. Remembering that sudden condensation in air or gas produces heat, and sudden rarefaction cold, and assuming that these alternations are so rapid that neither the heat nor the cold is conveyed to the other particles, within the volume considered, much beyond the point at which they originate, we see that this heat and cold will produce an elastic force of greater intensity than that in either of the other two cases; therefore the value of the velocity of propagation will be greater than that given in Eq. (214), which was deduced under the supposition of the simple ratio of the elastic force to the density expressed by $\frac{p}{d}$. It might be supposed that the influence of the heat produced in the condensation of the sound wave would be neutralized by the cold produced in the rarefaction, and that therefore the resultant effect would be zero. This, however, is not the case; for the heat in the condensation has increased the difference of elastic force between the condensed stratum and the one in its front, and hence has increased the velocity, while the cold in the rarefaction has caused an equal difference between the rarefied stratum and the one in rear, and has thus added an equal increment of velocity to this portion of the wave. This is true for each stratum affected by the sound wave. Hence the disturbance passes each stratum of the condensed and rarefied portions with the same velocity, and this may be regarded as the velocity of the wave.

189. Since the vibrational number of sound waves varies between 20 and 40000, for the extreme limits, the alternate condensations and rarefactions occur with sufficient rapidity to necessitate the application of the formula of Poisson for the determination of the velocity of sound in air and in other gaseous media.

190. ***Pressure of a Standard Atmosphere.*** Let p be the pressure of the atmosphere when the barometric column corresponds to 76 cm., the mercurial density being 13.5962, and g 981 dynes; we then have

$$p = 981 \times 13.5962 \times 76 = 1.01368 \times 10^6 \text{ dynes}, \qquad (218)$$

as the corresponding pressure of the atmosphere upon a square centimetre. But since the density of mercury, referred to the standard at the same locality, is independent of the locality, and hence independent of g, we may assume as the *standard atmosphere* that whose pressure on the square centimetre at all localities is equal to 10^6 dynes. Hence,

$$p = g \times d_m \times h = 10^6 \text{ dynes}. \qquad (219)$$

By substituting in this equation the value of g for the latitude of the place, and solving with reference to h, we will determine the barometric height corresponding to the standard atmosphere at that locality; g varies from 978.1 dynes at the equator to 983.11 dynes at the pole.

191. ***Height of the Homogeneous Atmosphere.*** If the atmosphere be supposed replaced by an atmosphere of uniform density D, as that of standard dry air at 0° C., and height H, exerting the same pressure, H may be obtained from the equation

$$p = g \cdot D \cdot H = 10^6 \text{ dynes}; \qquad (220)$$

from which we have

$$\left.\begin{aligned} H = \frac{10^6}{g \cdot D} &= \frac{10^6}{981 \times .0012759} = 7.9894 \text{ cm.} \times 10^5 \\ &= 7989.40 \text{ m.} = 26212.18 \text{ ft.}, \end{aligned}\right\} \qquad (221)$$

which is constant at the same locality, for the same temperature and barometric height. If the temperature become $\theta°$ C., we have, by the law of Charles or Gay-Lussac,

$$H' = (1 + \alpha\theta)\, H = H\,\alpha\,\tau, \tag{222}$$

in which τ is the absolute temperature, and α the coefficient of expansion.

192. Replacing the elastic force E by its equal, in terms of the homogeneous atmosphere, in Eq. (214), we have

$$V = \sqrt{\frac{E}{D}} = \sqrt{\frac{H\,\alpha\,\tau \cdot g \cdot D}{D}} = \sqrt{H\,\alpha\,\tau\,g}, \tag{223}$$

which is Newton's formula for the velocity of sound in air. Making $\tau = 273°$, corresponding to zero Centigrade, and $g = 981$ dynes, we have

$$V = \sqrt{7.9894 \times 10^5 \times 981} = 2.8 \times 10^4 = 280.0 \text{ metres.} \tag{224}$$

For any other temperature, we have

$$V = \sqrt{7.9894 \times 10^5 \times 981 \times \alpha\tau} = 280\sqrt{1 + \alpha\theta}. \tag{225}$$

193. These values of the velocity of sound in air are about one-sixth less than those determined by experiment, the discrepancy being due to the supposition that Mariotte's law expresses the relation of pressure and density. The law of Poisson is, however, applicable; hence we have

$$p' = p\left(\frac{D'}{D}\right)^{\gamma};$$

differentiating,

$$dp' = \gamma p \left(\frac{D'}{D}\right)^{\gamma - 1} \frac{dD'}{D},$$

$$\frac{dp'}{dD'} = \gamma\,\frac{p}{D}\left(\frac{D'}{D}\right)^{\gamma-1} = \gamma\,\frac{p'}{D'}. \tag{226}$$

Whence we see that when a sound wave is passing through air, the ratio of the increment of the elastic force to that of the density is equal to the ratio of the elastic force to the density, multiplied by the constant γ. The value of γ can be determined from a direct observation, by accurately measuring V, α, and θ, and substituting in the equation

$$V = \sqrt{\gamma \frac{p'}{D'}} = 10^4 \times 2.80 \sqrt{\gamma (1 + \alpha\theta)}, \tag{227}$$

and solving with respect to γ. Its value has been found to be, approximately, 1.41 for all simple gases not near their points of liquefaction. The final formula, therefore, is

$$\left.\begin{aligned} V &= 332.64 \text{ m.} \times \sqrt{1 + \alpha\theta} \\ &= 1091.35 \text{ ft.} \times \sqrt{1 + .003660°}, \end{aligned}\right\} \tag{228}$$

for the velocity of sound in air at the locality where $g = 981$ dynes, barometric height 76 cm., and temperature $\theta°$ Centigrade.

194. At West Point, assuming the barometric height to be 76 cm., and $g = 980.3$ dynes, we have, for the velocity of sound in air at any temperature,

$$\left.\begin{aligned} V &= \sqrt{980.3 \times 7.9894 \times 1.41 \times 10^5 \times \alpha\tau} \\ &= 332.3 \text{ m.} \times \sqrt{1 + \alpha\theta} \\ &= 1090.23 \text{ ft.} \times \sqrt{1 + \alpha\theta}. \end{aligned}\right\} \tag{229}$$

Since the value of $\alpha = \frac{1}{273}$, we see that the velocity increases nearly 2 feet for each degree Centigrade, and hence is greater in warm than in cold weather, all other things being equal. At 60° F., we may take the velocity of sound in air to be approximately 1123 feet per second.

195. The value of the velocity of sound in any gas can, in like manner, be obtained theoretically by substituting in the equation

$$V = \sqrt{\gamma \frac{p'}{D'}}, \tag{230}$$

for D' the density of the gas referred to that of air as unity, and for p' the value of the pressure in terms of the barometric height, γ being taken as 1.41; or it may be obtained more simply by dividing

$$V = 332.3 \text{ m.} \times \sqrt{1 + \alpha\theta} \tag{231}$$

by the square root of the density of the gas referred to air as unity.

At zero degrees Centigrade, we have for the theoretical value of the velocity of sound in the following gases:

Air,	332	Carbon dioxide,	262
Hydrogen,	1269	Carbon monoxide,	337
Oxygen,	317	Olefiant gas,	314

196. ***Velocity of Sound in Air and other Gases, as affected by their not being Perfect Gases.*** The formulæ of Mariotte, Charles, and Poisson are only applicable to perfect gases. This condition requires the elasticities to be perfect, and the excess of the elastic force which gives rise to wave propagation to be indefinitely small when compared with the elasticity of the gas in its quiescent state.

A series of experiments made by Regnault, the results of which are given in the Comptes Rendus, Vol. 66, page 209, show that these conditions are not fulfilled, and that the theoretical velocity therefore differs from the actual. The sounds were made in tubes of different cross-section, by discharging a pistol with different charges of powder. The results are grouped in the following table:

Diameter of Tube, 0.108 m. Length, 566.74 m.				Diameter of Tube, 0.3 m. Length, 1905 m.					Diam. of Tube, 1.10 m.	
Charge, 0.3 gr.		Charge, 0.4 gr.		Charge, 0.3 gr.		Ch'ge, 0.4 gr.	Charge, 1.5 gr.		Charge, 1.00 gr.	
Distances.	Mean Velocities.	Distances.	Mean Velocities.	Distances.	Mean Velocities.	Mean Velocities.	Distances.	Mean Velocities.	Distances.	Mean Velocities.
566.74	330.99	1351.95	329.95	1905	331.91	332.37	3810.3	332.18	749.1	334.16
1133.48	328.77	2703.00	328.20	3810	328.72	330.34	7620.6	330.43	920.1	333.20
1700.22	328.21	4055.85	326.77				11430.0	329.64	1417.9	332.50
2266.96	327.04	5407.80	*323.34				15240.0	328.96	2835.8	331.72
2833.70	327.52								5671.8	331.24
									8507.7	330.87
			* Uncertain.						11343.6	330.68
									14179.5	330.56
									17015.4	330.50
									19851.3	330.52

197. From these results we see: 1°, that the mean velocity of the same wave decreases from the origin; 2°, that it is less for the same charge and route in tubes of smaller diameter; 3°, that it

decrease less rapidly in tubes of larger diameter. Regnault also, by means of sensitive diaphragms, followed the course of the waves after they became inaudible, and obtained similar results with respect to these. He found that a sound produced by a pistol discharge, of one gramme of powder, became inaudible at distances of 1150, 3810, and 9540 metres, in tubes of 0.108 m., 0.30 m., and 1.10 m. diameter, respectively, and that the waves became insensible after traveling distances of 4056, 11430, and 19851 metres respectively. In the tube of 1.1 m. diameter, with a charge of 2.4 grains, the wave ceased to be audible at 58641 metres, and ultimately ceased at 97735 metres. These distances of audibility are, approximately, directly proportional to the diameters of the tube.

198. The mathematical theory discusses the case of a perfect gas, and assumes that the propagation in an indefinite tube is continuous. The above experiments show that this is not really the case. The assumptions made by implication in a perfect gas are:

1°. That the laws of Mariotte, Charles, and Poisson are true, but it is well known that no gas obeys exactly these laws.

2°. That its elasticity is unaffected by admixture with other gases.

3°. That the gas offers no opposition by its inertia to wave transmission; but experiment shows that an intense disturbance always produces a real motion of the surrounding particles, which increases the velocity, especially within sensible distances from the origin. Such is the case, no doubt, in cannon discharges, violent lightning-flashes, and other like instances.

4°. Theory supposes the excess of pressure due to a vibrating body small, in comparison with the quiescent barometric pressure; but in the cases cited above, the excess of pressure at the origin may be large, and hence cause an increase in the value of V near the origin. Therefore, the correction of Art. 193, called that of La Place, in such cases is not exact.

199. Regnault ascribes as the principal cause of the diminution of the intensity, the loss of kinetic energy by the reaction of the sides and ends of the tube, and confirms this by the fact that the sounds are quite audible outside the tube during their first passage, and in a less degree at each succeeding passage. As a secondary cause, he ascribes the influence of the walls of the tube in diminishing the elasticity without affecting the density. This is confirmed by the fact that in the above experiments, where the waves have been produced by the same charge, and hence have the same sensibility at the origin, they have not the same intensity after traveling over equal routes. The *mean* limiting velocity ought, therefore, to be the same, if the weakening is due to the loss of mv^2 on account of the sides. The experiments show that this is not the case; hence, the sides exercise another effect on air different from that indicated as the principal cause of the diminution of the intensity, an action affecting the elasticity and not the density. In free air this effect would be null, and in the tube of 1.1 m. it is taken as approximately so. The mean velocity of propagation, *in dry air at* 0° C., of a wave produced by the discharge of a pistol, and estimated from the origin to the point at which its sensibility can no longer be appreciated by the ear is, according to Regnault's experiments,

$$V = 330.6 \text{ m.}$$

The mean limiting velocity, considered from the origin to the point at which its existence can no longer be detected upon a sensitive diaphragm, is

$$V = 330.3 \text{ m.},$$

which differs from the mean limiting velocity in the 1.1 m. tube by only 0.32 m.

200. ***Velocity of Sound in Gases independent of the Barometric Pressure.*** Since an increase in the barometric pressure increases the elasticity and density in the same proportion, theory indicates that no change, due to this cause alone, will take place in the velocity. The experiments of Stampfer and Myrbach in the Tyrol, in 1822, between two stations whose difference in altitude was 1364 m., and of Bravais and Martins in Switzerland, in 1844, between two stations whose difference of level was 2079 m., indicated no variation in the velocity, due to the change in the barometric pressure. Regnault's experiments upon air in the tube 0.108 m. in diameter, over a distance of 567.4 m., with pressures varying from 0.557 m. to 0.838 m., and over a distance of 70.5 m., with pressures varying from 0.247 m. to 1.267 m., found no variation in the velocity, due to this cause.

The theoretical ratio of the velocities of sound in gases, given by

$$\frac{V'}{V} = \sqrt{\frac{D}{D'}}, \tag{232}$$

was experimentally confirmed to a near degree of approximation in the cases of hydrogen, carbon dioxide, and air. The tube 0.108 m., filled for a length of 567.4 m., gave for hydrogen 3.801 m., for carbon dioxide 0.7848 m., which differ but little from the theoretical values 3.682 m. and 0.8087 m., the velocity in air being taken as unity. Hence the formula may be taken as an expression for the limiting law. The determination of the velocity of sound in free air was made by means of reciprocal cannon discharges. There were two series of these experiments. For the first, consisting of 18 discharges, the membrane being 1280 metres distant, the mean velocity, referred to dry air at 0° C., was found to be

$$V = 331.37 \text{ m.}$$

For the second series, of 149 discharges, over a distance of 2445 m., during 11 days of trial, with the temperature of the air varying from 1.5° to 21.8° C., and with great variations in the wind, the mean velocity, referred to dry air at 0° C., was

$$V = 330.7 \text{ m.},$$

a sensible diminution of the velocity, due to the increased distance.

201. ***Velocity of Sound in Liquids.*** The value of the velocity of sound in liquids is likewise given by the general formula

$$\left.\begin{aligned} V &= \sqrt{\frac{gd_mH}{D\lambda}} = \sqrt{\frac{gd_mH}{\lambda} \times \frac{1}{D}} \\ &= \sqrt{E \times \frac{1}{D}} = \sqrt{\frac{E}{D}}, \end{aligned}\right\} \tag{233}$$

in which H is the arbitrary barometric height, d_m the density of mercury, and g the acceleration due to gravity. The numerator is then the pressure due to the height of the barometer, and when divided by λ, which is the diminution of the volume due to the increase of pressure, gd_mH gives the ratio of the pressure to the corresponding compression, and is therefore the measure of the elastic force of the medium. The square root of this quantity, divided by the square root of the density, will be the value of the velocity of sound in the liquid.

202. Colladon and Sturm made a series of experiments to determine the actual value of the velocity of sound in water, in Lake Geneva, in the year 1826. The sound was caused by the strokes of a hammer upon a bell submerged one metre below the surface, and so arranged that the epoch of the stroke could be determined by a flash of powder. The instant of hearing the sound was indicated by a stop-watch to within one-quarter of a second. The distance traveled by the sound was found to be 13487 m. to within 20 m., and the time of this travel, from a mean of many experiments, was found to be 9.4 s. The temperature of the water was 8.1° C., its density at that temperature, referred to that of water at the standard temperature, was unity plus a negligible fraction, its compressibility was taken at .0000495, and the barometric height at 76 cm. The density of mercury referred to the same temperature is 13.544, and $g = 9.8088$.

Making these substitutions in the preceding formula, we find

$$V = \sqrt{\frac{9.8088 \times 13.544 \times .76}{.0000495}} = 1428 \text{ m.}$$

The actual velocity found was $\frac{13487 \text{ m.}}{9.4} = 1435$ m., differing from the theoretical value but 7 m. The latter may itself vary within wider limits, on account of the inexactness of the value of the compressibility of water, whose most probably correct value, from the experiments of Regnault, is assumed to be .00004685.

203. The principal facts derived from these experiments of Colladon are (Tome XXXVI, Annales de Chimie) that at distances beyond 200 metres the quality of the sound is changed, and the sensation is similar to the quick, brief noise produced by the striking together of two knife-blades in air. The diminution of intensity with the distance is noticed, and at short distances, greater than 200 metres, it is not possible to tell whether the sound originates at a near origin of weak intensity, or at a distant origin with increased intensity. The duration is less than in air; as it should be from its value $\frac{\lambda}{V}$, λ being greater and V being smaller in air than in water. When the vibrations proceeding from the sounding body reach the surface of the water at great angles of incidence, the sound does not pass into the air. At distances greater than 400 to 500 metres, the ear in air does not hear the sound originating in the water. At 200 metres the sound is readily heard. In these experiments, the bell being placed 2 metres below the surface, the angle of incidence at 400 metres is approximately 89° 43′; at 200 metres, 89° 26′.

Finally, the existence of a sharper acoustic shadow shows that the wave lengths are proportionally shortened in water compared with the waves made in air by the same sounding body.

204. ***Velocity of Sound in Solids.*** The ordinary solids upon which experiments have been made for the determination of the velocity of sound are glass, the various metals, and wood. In the latter, from the manner of its growth in the tree, the three directions, along the axis, in the direction of the radius, and normal to the plane of these two, possess necessarily different elasticities. The coefficients of elasticity also differ in different species, and in the same species, when grown in different localities, under different circumstances of soil, temperature, and moisture. Reasonably exact determinations belong then only to the particular specimen experimented upon, and mean values are usually taken for any one kind of wood in a given direction. In metals and glass, variations of the coefficients arise from the methods of their manufacture, and modifications result from every circumstance which affects their density and other physical properties. None of the solids can be said to be perfectly homogeneous; but on the assumption that they are approximately so, different experimenters have obtained values for their coefficients which do not vary between very wide limits.

205. In solids, the sound may result either from transversal or from longitudinal vibrations. In the cases here considered, the vibrations are understood to be longitudinal, that is, the molecular displacements are in the direction of the propagation.

When a solid bar, taken as homogeneous, transmits a longitudinal vibration, the velocity of the propagation has been found to be given by the equation

$$V = \sqrt{\frac{g}{\lambda}}, \qquad (234)$$

in which λ is the elongation due to the weight of the bar. Substituting for λ its value in terms of Young's modulus,

$$\lambda = \frac{1}{E} \cdot \frac{Pl}{s}, \tag{235}$$

and making s equal to one square centimetre, l equal to one metre, and P the weight of the bar, we have

$$V = \sqrt{\frac{E}{D}}, \tag{236}$$

the same in form as has been found for gases and liquids.

206. Different methods have been employed to find E, viz., by the direct method of elongations or compressions, by flexure, by transversal and by torsional vibrations of the bar. The values given for the different metals, in Art. 23, have been obtained by Wertheim, by the method of elongations. Could we accurately determine the velocity of sound in solids by direct experiment, the value of E could be readily found by the solution of the above equation. But this velocity being very great compared with that in air, and because of the impracticability of finding sufficiently long homogeneous lengths, an accurate determination of E by this means is impossible. Biot, by a direct experiment on 951 metres in length of cast-iron pipe, found that the velocity was 10.5 times that in air; but the want of homogeneity, due to the numerous leaded joints, without doubt influenced this result appreciably. Wertheim found about the same value in wrought iron, by experimenting upon 4067.2 metres of telegraph wire.

207. Assuming the experimental values for E given in Art. 23, and taking g to be 981 dynes, the velocities of sound are, by the above formula, found to be as follows :

	E.	*D.*	V in Centimetres.	Ratio to V in Air.
Lead, . .	$177 \times 981 \times 10^6$	11.4	1.23×10^5	3.7
Gold, . .	$813 \times 981 \times 10^6$	19.0	1.74×10^5	5.3
Silver, . .	$736 \times 981 \times 10^6$	10.5	2.61×10^5	8.0
Copper, . .	$1245 \times 981 \times 10^6$	8.6	3.56×10^5	10.7
Iron, . .	$1861 \times 981 \times 10^6$	7.0	5.13×10^5	15.5
Steel, . .	$1955 \times 981 \times 10^6$	7.8	4.99×10^5	15.0

For glass, with density of 2.94, V has been found to be, by the same method, 4.53 cm. $\times 10^5$; and for brass, of density of 8.47, $V = 3.56$ cm. $\times 10^5$; or 13.6 and 10.8 times the velocity in air, respectively.

208. The following velocities of sound in wood, deduced from the observations of Wertheim and Chevandier (Comptes Rendus, 1846), are taken from "Everett's Physical Constants," page 65, from which also several of the above numbers have been obtained:

	Along Fibre.	Radial.	Tangential.
Pine, ...	3.32×10^5	2.83×10^5	1.59×10^5
Beech, ...	3.34×10^5	3.67×10^5	2.83×10^5
Birch, ...	4.42×10^5	2.14×10^5	3.03×10^5
Fir, ...	4.64×10^5	2.67×10^5	1.57×10^5

209. The preceding values of the velocities of sound in solids are true only when the medium is in the form of a bar of small cross-section. Wertheim has shown by his investigations, based on the theory of Cauchy, that the corresponding velocities in extended homogeneous solids are greater than the above results in the ratio of

$$\sqrt{\frac{3}{2}} \; : \; 1.$$

210. ***Reflection and Refraction of Sound.*** The laws deduced in Art. 77 for the reflection and refraction of wave motion are applicable to the undulations of sound. From the equation

$$\sin \phi = \mu \sin \phi', \tag{237}$$

the direction of any deviated ray or that of any deviated plane wave by a plane surface, can be found when $\frac{V}{V'}$ is substituted for μ. If $V' > V$, then $\phi' > \phi$, and the refracted ray is thrown from the normal; conversely, if $V' < V$, then $\phi' < \phi$, and the refracted ray is bent towards the normal. A ray of sound in air, incident on the surface of water, will be refracted, provided the angle of incidence be less than 13° 26′; for since V' in water is about 1428 m., and V in air about 332, we have

$$\mu = \frac{332}{1428} = .2325,$$

and $\sin \phi = .2325,$ or $\phi = 13° 26'.$

For greater incidences the ray is totally reflected, and does not enter the water.

211. *Consequences of the Laws of Reflection.*

1°. If a sound originate at one of the foci of an ellipsoid, it will be reflected to the other focus.

2°. If at the focus of a paraboloid, the rays of sound will be reflected in lines parallel to the axis, and can be again collected at the focus of another similar paraboloid, with sensibly undiminished intensity. The slightest sound, as the ticking of a watch, may be employed to illustrate this case of reflection.

3°. The speaking-trumpet and speaking-tube are employed to prevent the too rapid dissipation of sound. The former, partly by reflection from its sides and largely by resonance, concentrates the sound within the volume of the cone whose apex is the mouth-piece and whose section is that of the other end of the trumpet. The speaking-tube confines the energy in the narrow compass of the tube, the loss being insignificant in the ordinary lengths employed.

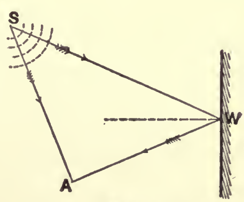


Figure 28.

4°. When a sound is reflected by any obstacle which prevents its direct transmission, and the observer is at such a distance that the direct and reflected sounds are not confounded, the reflected sound is called an *echo*. Thus, if A be the position of the observer, S the origin from which a sound of short duration emanates, and W the obstacle, such as a wall, then the direct sound will reach the observer in the time $\frac{SA}{332.4}$, and the reflected sound in the time $\frac{SW + WA}{332.4}$, the temperature being 0° C. If $\frac{SW + WA - SA}{332.4}$ be sufficiently great, so that the reflected sound arrives after the cessation of the direct sound, then the echo will be heard, provided the intensity be of sufficient value. If the two sounds commingle, the resultant sound will be prolonged, and partial resonance will ensue. The number of distinct impressions distinguished by the ear will determine the shortest difference of route necessary to establish the echo. Thus, if we take nine per second, $\frac{332}{9}$ or 37 m. is the shortest difference of route at 0° C.

5°. The conditions of interference of sound are the same as those discussed in Arts. 65–68. Hence, it is theoretically possible that two sounds affecting the ear simultaneously will result in silence, and practically it will be shown, in the lectures on this part of the course, that such an experiment is also possible. Other illustrations of interference are also reserved for the lectures.

212. ***Refraction of Sound.*** In order that the rays of sound shall converge after deviation by refraction, we see from the formula that $\mu = \frac{V}{V'}$ must be greater than unity. Then the deviated wave will, in general, become converging, and the energy accumulate on an ever decreasing surface. Examining the table, Art. 195, we see that V' in carbon dioxide is 262 m., and hence, when the incident medium is air,

$$\mu = \frac{332}{262} = 1.25,$$

and

$$\sin \phi = 1.25 \sin \phi'. \tag{238}$$

The sound lens devised by Sondhaus is a double convex lens of collodion filled with carbon dioxide, which collects the sound rays proceeding from any sonorous body and concentrates them appreciably at another point on the opposite side of the lens. By means of a concave lens of the same material, filled with hydrogen, $V' = 1269$ m., it will be evident, after the study of the properties of lenses, as explained in optics, that a similar result would be effected. The slight noise produced by the ticking of a watch may be collected by this means at a point so that the noise is audible, when without this assistance it would be inappreciable at the same point.

213. ***General Equations for the Vibratory Motion of a Stretched String.*** The bodies usually employed to produce musical sounds by their vibrations are strings, rods air-columns, plates, bells, etc. When the vibrations of the particles are perpendicular to the direction of wave propagation, they are called *transversal,* and when in the same direction, *longitudinal.*

We will first consider the vibrations of a perfectly elastic and flexible string, supposed to be stretched between two points whose distance apart is l, by a force which produces a tension T. Let the elongation be that given by

$$l' - l = \frac{T}{E} l, \tag{239}$$

in which l is the natural length, l' the length after the tension T is applied, and E is the longitudinal modulus. If the displacements of the string from its position of rest be due to the incessant action of forces whose rectangular accelerations are $X, Y, Z,$ these with the tension T will be the only extraneous forces considered.

Let m be the mass of any element; $x,\ y,\ z,\ x + dx,\ y + dy,\ z + dz,$ the co-ordinates of its extremities and its length ds; α the area of its cross-section, and ρ its density; then

$$m = \rho\alpha\, ds.$$

Let the components of T at $x, y, z,$ be

$$T\frac{dx}{ds}, \qquad T\frac{dy}{ds}, \qquad T\frac{dz}{ds};$$

and at $x + dx,\ y + dy,\ z + dz,$ be

$$T\frac{dx}{ds} + dT\frac{dx}{ds}, \qquad T\frac{dy}{ds} + dT\frac{dy}{ds}, \qquad T\frac{dz}{ds} + dT\frac{dz}{ds}.$$

The general equations of motion will then be

$$\left.\begin{aligned} \rho\alpha\, ds\left(X - \frac{d^2x}{dt^2}\right) + dT\frac{dx}{ds} &= 0, \\ \rho\alpha\, ds\left(Y - \frac{d^2y}{dt^2}\right) + dT\frac{dy}{ds} &= 0, \\ \rho\alpha\, ds\left(Z - \frac{d^2z}{dt^2}\right) + dT\frac{dz}{ds} &= 0. \end{aligned}\right\} \tag{240}$$

214. These equations are simplified when we suppose that the string is arbitrarily displaced from its position of equilibrium, and abandoned to itself, without the action of the forces X, Y, Z. It will then oscillate about its position of rest, and the only extraneous force that acts will be the tension T, whose intensity will vary between known limits. Let the axis of x coincide with the string in its position of rest, and the co-ordinates of the element m, at the time t, be $x + \xi$, η, ζ. If the displacement be supposed small, ξ, η, and ζ are functions of x and t, and x is independent of t, and the above equations reduce to

$$\left.\begin{aligned} \rho\alpha\, dx \frac{d^2\xi}{dt^2} - dT \frac{d(x+\xi)}{ds} &= 0, \\ \rho\alpha\, dx \frac{d^2\eta}{dt^2} - dT \frac{d\eta}{ds} &= 0, \\ \rho\alpha\, dx \frac{d^2\zeta}{dt^2} - dT \frac{d\zeta}{ds} &= 0. \end{aligned}\right\} \qquad (241)$$

Let T' be the tension when the string is straight, and T when the string is displaced; the length of the element is in the first case dx, and in the second ds; these are connected by the equations

$$ds = dx\left(1 + \frac{T - T'}{E}\right), \qquad (242)$$

$$ds^2 = (dx + d\xi)^2 + d\eta^2 + d\zeta^2; \qquad (243)$$

from which, when $d\eta$ and $d\zeta$ are very small, we have

$$ds = dx + d\xi, \qquad (244)$$

$$T = T' + E\frac{d\xi}{dx}. \qquad (245)$$

Substituting in Eqs. (241), we have

$$\left.\begin{aligned} \rho\alpha \frac{d^2\xi}{dt^2} &= E\frac{d^2\xi}{dx^2}, \\ \rho\alpha \frac{d^2\eta}{dt^2} &= T'\frac{d^2\eta}{dx^2}, \\ \rho\alpha \frac{d^2\zeta}{dt^2} &= T'\frac{d^2\zeta}{dx^2}, \end{aligned}\right\} \qquad (246)$$

Replacing $\frac{E}{\rho a}$ and $\frac{T'}{\rho a}$ by u^2 and v^2 respectively, we have

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= u^2\frac{d^2\xi}{dx^2},\\ \frac{d^2\eta}{dt^2} &= v^2\frac{d^2\eta}{dx^2},\\ \frac{d^2\zeta}{dt^2} &= v^2\frac{d^2\zeta}{dx^2}.\end{aligned}\right\} \quad (247)$$

The integration of these three partial differential equations give (Analytical Mechanics, Appendix IV),

$$\left.\begin{aligned}\xi &= F(x + ut) + f(x - ut),\\ \eta &= F(x + vt) + f(x - vt),\\ \zeta &= F(x + vt) + f(x - vt).\end{aligned}\right\} \quad (248)$$

215. The first equation determines the longitudinal vibrations, or those along the axis of the string, and the other two give the transversal vibrations along y and z respectively. Because of the independence of the differential equations, the three vibrations in general coexist and are wholly independent of each other, and since the differential equations are of the same form, we see that the two kinds of vibrations are subjected to the same laws. They may each be discussed separately. Each is due to a progressive motion forward and backward along the string. These motions may be of the most varied character, but the particular form of the motion depends on the form of the functions whose symbols are F and f. The only conditions imposed so far are that for $x = 0$ and $x = l$, ξ, η, and ζ are zero for all values of t. These, together with any assumed initial conditions, will enable us to determine the form of the functions F and f, and thus complete the solution of the problem.

216. Since the vibrations parallel to y and z are exactly alike in every particular, the discussion of one will do for the other, and we will consider that of y, given by the equation

$$\eta = F(x + vt) + f(x - vt). \quad (249)$$

Assume the conditions that at the epoch, or when $t = 0$,

$$\eta = \phi(x) \quad \text{and} \quad \frac{d\eta}{dt} = v\psi'(x), \tag{250}$$

in which the functions ϕ and ψ are supposed known, and that ψ' is the derived function of ψ. If $t = 0$, we have

$$\eta = \phi(x) = F(x) + f(x), \tag{251}$$

$$\frac{1}{v} \cdot \frac{d\eta}{dt} = \psi'(x) = F'(x) - f'(x); \tag{252}$$

$$\therefore \quad \psi(x) = F(x) - f(x); \tag{253}$$

and hence,

$$F(x) = \frac{\phi(x) + \psi(x)}{2}, \tag{254}$$

$$f(x) = \frac{\phi(x) - \psi(x)}{2}. \tag{255}$$

Therefore $F(x)$ and $f(x)$ are known for all values of x from 0 to l, when, as is supposed, $\phi(x)$ and $\psi(x)$ are known between the same limits.

For the extremities, we have, by placing $x = 0$ and $x = l$,

$$F(vt) + f(-vt) = 0, \tag{256}$$

$$F(l + vt) + f(l - vt) = 0; \tag{257}$$

whence, $F(vt)$ and $f(-vt)$ are equal, with contrary signs, and thus become known for all values from $t = 0$ to $t = \infty$.

217. The value of η can be expressed by means of a single function by substituting $vt + l - x$ for vt in Eq. (257); whence,

$$-F(2l - x + vt) = f(x - vt); \tag{258}$$

which in Eq. (249) gives

$$\eta = F(x + vt) - F(2l - x + vt). \tag{259}$$

Again, for vt, in Eq. (257), substitute $l + vt$; then

$$F(2l + vt) = -f(-vt) = F(vt); \tag{260}$$

whence we conclude that the function F takes the same value when the variable vt is increased by $2l$; and therefore by $4l$, $6l$, $8l$, ...

or $2nl$, n being a positive whole number. Therefore, if $F(vt)$ is known from $vt = 0$ to $vt = 2l$, its value is known for all values from $t = 0$ to $t = \infty$.

Replace vt by $l - vt$, in Eq. (257), vt being less than l; then

$$F(2l - vt) = -f(vt); \tag{261}$$

but $f(vt)$ is known for all values of vt between 0 and l; therefore $F(vt)$ is known for all values of vt between l and $2l$.

Hence, the value of $F(x + vt)$ is known for all values of $x + vt$ from 0 to ∞; and, similarly, the value of $f(x - vt)$ can be found for all values between 0 and $-\infty$; and therefore the problem is completely solved.

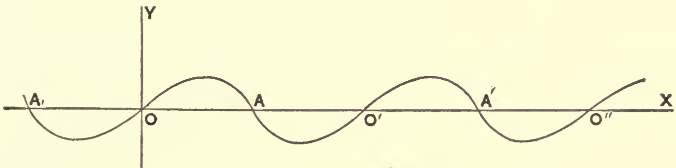


Figure 29.

218. The function whose symbol is F is subject to the following conditions, derived from Eqs. (256), (257), (260), (261),

$$F(x) = -F(-x), \tag{262}$$

$$F(l + x) = -F(l - x), \tag{263}$$

$$\left.\begin{aligned} F(x) = F(2l + x) = F(4l + x) = \ldots\ldots\ldots \\ = F(2nl + x), \end{aligned}\right\} \tag{264}$$

$$F(x) = -F(2l - x) = -F(4l - x) = \ldots. \tag{265}$$

From Eq. (262) we see that the curve represented by $\eta = F(x)$ is continued in similar forms on each side of O in the figure; from Eq. (263), that the forms are similar on each side of A; from Eq. (264), that the form is repeated from O′ to O″ exactly as from O to O′; and from Eq. (265), that the form of the curve inverted is the same from O′ to A as from O to A.

The motion of any particle is that of oscillation about its place of rest, and of which the period is $\frac{2l}{v}$. This vibratory motion is gradually diminished, while the period remains unchanged, because of the energy communicated to the air, and through the points of attachment to other bodies. The time of one complete oscillation is

$$t = \frac{2l}{v} = 2l\sqrt{\frac{\rho\alpha}{T'}}, \tag{266}$$

and the number of oscillations in the unit of time is

$$n = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T'}{\rho\alpha}}. \tag{267}$$

Therefore in the transversal vibrations of a string, the resulting pitch is inversely proportional to its *length*, directly as the square root of the *tension* when straight, and inversely as the square root of the *density* by the *area of cross-section.*

219. The number of longitudinal oscillations in the unit of time is

$$n' = \frac{u}{2l} = \frac{1}{2l}\sqrt{\frac{E}{\rho\alpha}}, \tag{268}$$

whence the pitch depends only upon the *length* of the string and the *material* of which it is made, and is independent of the tension, unless the latter should be so considerable as to change the value of E. Experiment appears to indicate that the longitudinal pitch increases slightly with the tension; but this may be accounted for in the elongation experienced, which is always accompanied with a slight diminution of density ρ, and should this occur, the formula indicates that the pitch should rise.

The ratio of the numbers for the same string is given by

$$\frac{n'}{n} = \sqrt{\frac{E}{T'}} = \sqrt{\frac{l}{\Delta l}}. \tag{269}$$

M. Cagniard Latour experimented on a cord of 14.8 m. in length, and found

$$\frac{n'}{n} = \frac{188}{7} \quad \text{and} \quad \Delta l = 0.05 \text{ m.}$$

Substituting in the formula, we have

$$\frac{188}{7} = \sqrt{\frac{14.8}{\Delta l}};$$

whence, $\Delta l = 0.052$ m., a sufficiently near approximation.

220. The preceding values of n and n' are the least numbers of transversal and longitudinal vibrations of the string, and therefore correspond to its fundamental tones; but we know that each of the vibrations is decomposed into any number of vibrations of equal periodicity, when the string is divided into a like number of symmetrical parts. This can be shown more readily when the integral equation is expressed in a series which is a function of sines and cosines. Thus, it is evident that a possible solution of the differential equation

$$\frac{d^2\eta}{dt^2} = v^2 \frac{d^2}{dx^2}, \tag{270}$$

is given by

$$\eta = \left(A_i \cos \frac{i\pi vt}{l} + B_i \sin \frac{i\pi vt}{l}\right) \sin \frac{i\pi x}{l}, \tag{271}$$

when the conditions with respect to the extreme points are unchanged. In this equation, i is any entire positive number which marks the order of the term, and A_i, B_i, are constant coefficients depending on i and on the initial state of the string. If then this state is such that η is constant only for the terms for which i is a multiple of another entire number n, the string will return to the same state at the end of each interval of time $\frac{2l}{nv}$, which is the duration of its similar and isochronous vibrations. Under this supposition, the $n - 1$ points of the curve corresponding to distances

$$x = \frac{l}{n}, \quad = \frac{2l}{n}, \quad = \frac{3l}{n}, \quad = \text{etc.},$$

will be nodes, that is, will remain at rest during the whole period of the motion.

Since the value of η is linear, every value corresponding to $i =$ 1, 2, 3, 4, etc., will be a solution, and the sum of all the values of η will also be a solution of the differential equation; hence we will have for the general integral equation

$$\eta = \Sigma_{i=1}^{i=\infty}\left(A_i \cos \frac{i\pi vt}{l} + B_i \sin \frac{i\pi vt}{l}\right) \sin \frac{i\pi x}{l}. \tag{272}$$

221. The values of A_1B_1, A_2B_2, etc., are in general arbitrary, and we may suppose all to vanish up to any order n, while the rest remain arbitrary. If A_1B_1 are not zero, there are no actual nodes except the fixed ends, and the first simple tone is that whose period is τ and whose wave length is $2l$. If there is one node, the period is $\frac{\tau}{2}$, and the gravest simple tone is that of wave length l; and, generally, if there are $n-1$ nodes, the period is $\frac{\tau}{n}$, and the gravest tone is the $(n-1)^{th}$ harmonic of the fundamental tone.

When the string vibrates without nodes, the series of harmonic tones is in general complete, and a practised ear can distinguish ten or more. It is also possible to make a string vibrate in such a manner that for any proposed value of n the coefficients $A_n B_n$, $A_{2n} B_{2n}$, etc., shall disappear, so that the component harmonic vibrations whose periods are $\frac{\tau}{n}$, $\frac{\tau}{2n}$, etc., are extinguished. When this is done, the ear does not distinguish these tones, and we may therefore conclude, from what precedes, that each component tone actually heard is produced by the corresponding harmonic vibration of the string.

222. The same general method may be applied to the longitudinal vibration of a rod, and the differential equation will be, as in the case of the longitudinal vibration of a string, of the form

$$\frac{d^2\xi}{dt^2} = V^2 \frac{d^2\xi}{dx^2}, \tag{273}$$

of which the integral equation is

$$\xi = F(x + Vt) + f(x - Vt), \tag{274}$$

and which may be put under the form of

$$\xi = x + \Sigma \cos \frac{i\pi x}{l}\left(A_i \cos \frac{i\pi Vt}{l} + B_i \sin \frac{i\pi Vt}{l}\right), \tag{275}$$

in which ξ is the distance from the fixed origin at any time t to the particles in a plane section of the rod, of which the natural distance from the end of the rod is x. The value of x therefore depends only on the particular section considered, and is independent of the origin of ξ; but if the vibrations cease, the periodic part of Eq. (275) would vanish, and we would have $\xi = x$ for all points of the rod, and therefore the periodic part gives the displacement $(\xi - x)$ at the time t of the section determined by the value of x.

The periodic part does not in general vanish for any value of x, so that there are in general no nodes. But there will be n nodes at sections for which x is any odd multiple of $\frac{l}{2n}$, provided A_i, B_i, vanish for all values of i except odd multiples of n. Thus the rod may have any number of nodes, of which those next the ends are distant from the ends by half the distance between any two consecutive nodes.

223. Differentiating Eq. (275), we have

$$\frac{d\xi}{dx} = 1 - \frac{\pi}{l}\Sigma i \sin\frac{i\pi x}{l}\left(A_i \cos\frac{i\pi Vt}{l} + B_i \sin\frac{i\pi Vt}{l}\right); \quad (276)$$

which, when $x = 0$ and $x = l$, becomes

$$\frac{d\xi}{dx} = 1. \quad (277)$$

But

$$\frac{d\xi}{dx} = \frac{\rho'}{\rho},$$

in which ρ' is the natural density, and ρ is the changed density. We see, therefore, that there is no change of density at the free ends. If A_i, B_i, vanish, except where i is a multiple of n, the variable part of $\frac{d\xi}{dx}$ vanishes when x is a multiple of $\frac{l}{n}$. Hence, where there are nodes, the sections in which there is no variation of density are those which bisect the nodal intervals in the state of equilibrium, and these sections of no variation of density are also sections of greatest displacement, since, Eq. (275), $\cos\frac{i\pi x}{l}$ is equal to ± 1 for values of x which make $\sin\frac{i\pi x}{l} = 0$.

224. The vibration represented by Eq. (275) consists of an infinite number of simple harmonic vibrations, each of which might subsist by itself; the n^{th} component would have n nodes, and its period would be $\frac{2l}{nV}$, the period of the fundamental tone being $\frac{2l}{V}$; therefore the wave length is twice the length of the rod. For the general case in which there is a node at the middle of the rod, $\cos\frac{i\pi x}{l}$, vanishes for all values of i when $x = \frac{l}{2}$. Then A_i, B_i, must vanish for all even values of i. The gravest tone is then the fundamental tone of the rod, and the higher tones of even orders disappear. The first upper tone will be a twelfth above the fundamental. In this case, the middle section might become absolutely fixed, and either half be taken away without disturbing the motion, so as to leave a rod of half the length, with one free and one fixed end. Therefore the fundamental tone of a rod with one end fixed is the same as that of a free rod of twice the length. The wave length is then four times the length of the rod, and the even orders of the harmonics are wanting.

225. The vibrations of air columns are theoretically the same as that of a free rod or one fixed at an end, and the same conclusions, modified by the elasticity of the air and its velocity of wave propagation, will theoretically apply. We will, however, determine the positions of the nodes and ventral segments of vibrating air columns in a simpler manner.

226. ***Vibrations of Air Columns.*** We will first suppose a single sonorous pulse moving in an air column, and consider, 1°, the column closed at one end and open at the other. Each stratum passes through all changes of density during the periodic time τ, while the pulse moves a distance λ; the air particles describe longitudinal vibrations, whose amplitudes depend on the intensity of the sound. When the condensation, which we suppose is in advance, reaches the closed end, the air stratum at that place, not having freedom of motion, undergoes changes of density alone. These changes are each immediately reflected in succession, and the condensation moves from the closed end with the same velocity with which it would have proceeded beyond had there been no obstruction to its progress. Hence we see that at the instant the rarefaction first reaches the closed end the reflected condensation affects the same strata as the incident rarefaction, and disregarding the loss due to incidence, the air strata will, at this instant, in the length $\frac{\lambda}{2}$ from the closed end, have their normal density throughout. The velocities of the air particles, at the same instant, will likewise be that compounded algebraically of those belonging to the reflected condensation and incident rarefaction. Now when a sonorous body is vibrating, the sound undulations follow each other periodically, and therefore the reflected and incident pulses will be distributed throughout the column. The densities and motions of the strata will therefore result from the combination of the same elements in the incident and reflected pulses.

227. Let the curve $rm''mA$ represent the direct wave at any instant, and its ordinates the corresponding compressions and dilatations of the air on the line $n'b'$ due to this wave; the curve $m''m'bA$ and its ordinates will, in like manner, represent the reflected wave from the stopped end AA'.

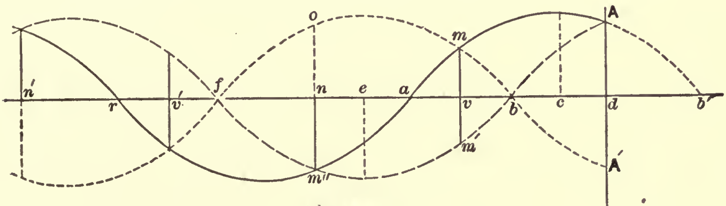


Figure 30.

We see that at points such as v, v', etc., at $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$, etc., from AA', the condensations or dilatations due to the direct wave will always be contrary and equal to the dilatations or condensations due to the reflected wave; hence, at these points, the normal density of the air will ever exist. But at points such as n, n', etc., at distances of $\frac{1}{2}\lambda$, λ, $\frac{3}{2}\lambda$, etc., from AA', the condensations or dilatations of each are of equal value, and of the same kind, and exist simultaneously; therefore the resultant condensation or dilatation is double that due to either. At these points then the air undergoes all variations of density during the period τ. The density at all points from n to v and to v', undergoes decreasing variations from the maximum at n to zero at v and v'.

228. With regard to the velocities of the air particles at different distances from AA', since the motions of the particles change direction abruptly at reflection, the ordinates of the curve $A'bmo$ will represent the velocities due to the reflected wave, and those of $Amm''r$ may now represent those of the direct wave. Then at v, v', etc., the velocities are zero only at instants separated by $\frac{\tau}{2}$, and at all other times have values that vary from zero to that represented by double the maximum ordinate; at n, n', etc., the velocities are always zero, and therefore the air at these points is quiescent, while undergoing changes of density. At intermediate points, both changes in velocities and density occur.

Hence, we conclude that *nodes* will be developed in a column of air closed at one end, when it is traversed by a sonorous wave, at distances from the stopped end of 0, $\frac{2\lambda}{4}$, $\frac{4\lambda}{4}$, $\frac{6\lambda}{4}$, etc.

The vibrating parts between the nodes are called *ventral segments*, and their middle points are at distances of $\frac{\lambda}{4}$, $\frac{3\lambda}{4}$, $\frac{5\lambda}{4}$, etc., from the stopped end.

229. 2°. ***Open Air Columns.*** Let the two tubes AM and MB, of unequal diameter, be united at M, and admit that there is no abrupt change of density of the air at M. The consequence of a contrary supposition is that the opposite sides of the infinitely thin stratum M would be subjected to unequal pressures, whose finite difference wonld generate in M an infinite velocity in a finite time. Hence, the density has the property of continuity in its variation throughout AB. It is not essential that the variation of the velocities of the particles of air should be continuous, nor is it incompatible with this condition.

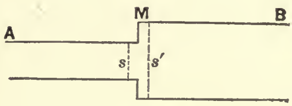


Figure 31.

Let s and s' be the areas of sections in AM and MB, indefinitely near M; v and v' the velocities of the air particles in s and s', at the time t; then $vs\,dt$ and $v's'\,dt$ will be the volume of air passing s and s' during dt, and $(vs - v's')\,dt$ will be the increment of the quantity of air in the volume ss' in the time dt, which will be proportional to the increase of density in ss'. But in order that the increment of density may be compatible with the supposed continuity of the pressure, it is evident that $(vs - v's')\,dt$ must be an infinitesimal of the second order, and equal to zero when s and s' are coincident. Hence, at the limit we have

$$vs = v's'\,;$$

therefore, there will be a wave propagated in MB, whose intensity, determined by the value of v', will become more and more inappreciable as s' becomes greater and greater than s. Let MB be increased indefinitely in area, as when the tube AM opens into the external air, then v' becomes very small, and the transmitted wave becomes negligible, as is the case in open pipes. There will then be a reflected wave in AM, composed of a rarefaction followed by a condensation, when the direct wave is a condensation followed by a rarefaction. The velocities of the air particles will then be theoretically equal in value, and the same in direction in the two waves. The curves of Fig. 30 will illustrate the case of open pipes, if A$'bof$ represent the densities and A$bm''f$ the velocities of the air particles in the reflected wave. The nodes and middle points of the ventral segments will then be at distances of

$$\frac{\lambda}{4}, \quad \frac{3\lambda}{4}, \quad \frac{5\lambda}{4}, \quad \frac{7\lambda}{4}, \quad \text{etc.,}$$

and

$$0, \quad \frac{2\lambda}{4}, \quad \frac{4\lambda}{4}, \quad \frac{6\lambda}{4}, \quad \text{etc.,}$$

from M, the open end of the tube, respectively.

230. These laws, which determine the positions of the nodes and ventral segments of vibrating air columns, are known as Bernouilli's laws. From them we see that the harmonics of open pipes are in the order of the natural numbers, and that those of closed pipes are as the odd numbers. Thus, the open pipe can give, by an increased pressure, the octave, the twelfth, the fifteenth, etc., while the closed pipe gives the twelfth, the seventeenth, etc. Experiments with organ-pipes verify the laws of Bernouilli only approximately ; that is, that the nodes are not exactly at the positions defined above, nor are the nodes exactly places of rest. Organ-pipes are usually made to speak by forcing a current of air through a narrow slit, and causing it to impinge against a thin lip. Of the many vibratory motions produced in this manner, there is always one whose periodicity is such that, by the resonance of the pipe, its intensity will be raised to such a degree as to produce a marked and determinate musical sound, called the fundamental tone of the pipe. Other vibratory motions, which undoubtedly exist, are either destroyed by the interference of the reflected waves, or have so feeble an intensity as to be negligible. The wave length of the fundamental tone is, as we have seen above, double the length of the open pipe, or four times the length of the closed pipe, approximately. The discrepancy between experiment and theory arises from the fact that the hypothesis is not in accord with what actually occurs in the pipe. Without considering these minutely, it is sufficient to note the perturbations at the embouchure by the air current, the modifications in the pipes by the moving air, and the induced vibrations of the material of the pipe at the sides and closed end, to account for the greater discrepancies.

231. ***Relative Velocities of Sound in Different Material.*** Since in any medium, we have $\lambda = V\tau = V\frac{1}{n}$, in which n is the vibrational number for a note of definite pitch, λ the corresponding wave length in the same medium, and V the velocity of sound, it is readily seen that if free rods of different material be taken, of such lengths as to give the same note when put into longitudinal vibration, we will have

$$\lambda = V\frac{1}{n}, \qquad \lambda' = V'\frac{1}{n}, \qquad \lambda'' = V''\frac{1}{n}, \quad \text{etc.};$$

whence

$$\lambda : \lambda' : \lambda'' :: V : V' : V''.$$

But as $\frac{\lambda}{2}$, $\frac{\lambda'}{2}$, $\frac{\lambda''}{2}$, etc., are the lengths of free rods that give the fundamental tone, we see that such lengths are directly proportional to the velocities of sound in the several media, when their lengths are great compared to their cross-sections. Knowing then the velocity of sound in any material, we can by experiment find that in others by this method. Then having the velocities, we can by substitution in the formula

$$V = \sqrt{\frac{E}{D}},$$

find the value for the longitudinal modulus E.

232. Applying the same principle to any gas and comparing the velocity in it with that in air, by the formula

$$V = \sqrt{\frac{gh\Delta}{d}[1 + \alpha\theta]\gamma}, \tag{278}$$

the values of γ, or the ratio of its specific heats, can be readily obtained. By this means Dulong found the following results :

	DENSITY.	VELOCITY.	$\gamma = \frac{c}{c_1}$.
Air	1.	333.	1.421
Oxygen	1.1026	317.7	1.415
Hydrogen	0.0688	1269.5	1.407
Carbon Dioxide	1.524	261.6	1.338
Carbon Monoxide, ...	0.974	337.4	1.427

Under the assumption that the gas is perfect, simple, and far from its point of liquefaction, γ is assumed to have the constant value of 1.41. The above results show that this value should be considered as the limit to which γ approximates and only reaches under the particular suppositions made.

233. *Transversal Vibration of Elastic Rods.* An elastic rod is a rigid body whose cross-section, considered uniform throughout, is taken as very small compared with its length. The rod or bar may be arranged in six different ways, depending on the method by which its ends are sustained, viz.:

1°. The rod may be free at both ends.
2°. It may be firmly fixed at both ends.
3°. It may be fixed at one end and free at the other.
4°. It may be supported at one end and free at the other.
5°. It may be fixed at one end and supported at the other.
6°. It may be supported at both ends.

It may yield its fundamental tone by vibrating as a whole, or give tones of higher pitch by dividing itself into vibrating parts separated by nodes. The formula

$$N = \frac{n^2 t}{2l^2}\sqrt{g\frac{E}{D}} \tag{279}$$

gives the number of vibrations in all cases, as has been verified by experiment. In this formula, N is the number of vibrations per second; n a constant depending on the manner in which the rod is arranged at the ends and on the number of nodes formed ; t is the thickness, measured in the plane of vibration; l is the length, E the rigidity, and D the density of the rod.

234. This formula shows that the vibrational number is independent of the width, provided it be small as at first supposed; that it is directly proportional to the thickness, inversely as the square of the length, and directly as the square root of the rigidity divided by the density.

1°. The rod is free at both ends. Lissajous has determined by careful experiments that the following formulæ apply, viz.:

$$d = \frac{2l}{2n-1}, \qquad s' = \frac{5l}{2(2n-1)}, \qquad s = \frac{0.6608l}{2n-1}, \quad (280)$$

in which l is the length, n the number of nodes formed, d the distance between two consecutive nodes, s the distance from the free ends to the nearest nodes, and s' the distance from the free ends to the second nodes. Hence from these formulæ, we see that the intermediate nodes are equidistant; that the distance from the extreme nodes to the next adjacent is nearly 0.92 of the distance between two consecutive intermediate nodes; that $s : s' :: 0.2643 : 1$, and $s : d :: 0.33 : 1$. Experiment confirms these results whatever be the number of the nodes. The positions of the nodes are made visible by sprinkling sand on the bar, and noticing the lines on which it accumulates when the bar or rod is put in vibration.

2°. Both ends are fixed. When the ends are so fixed as not to modify its elasticity at these points, it can vibrate freely, and the nodes are found to be located at the same places as in a free rod of the same length, except that the extreme nodes are at the fixed ends. The first two of formulæ (280) are then applicable to this case.

3°. The rod is fixed at one end and free at the other. There will then be 0, 1, 2, 3, ... nodes depending upon the manner by which it is put in vibration. If the fixed end be regarded as a node, the first of the above formulæ is applicable, and the other two apply to the free end only. Therefore these first three cases are all reducible with the modifications mentioned to that of a free rod at both ends.

4° and 5°. In these cases the supported end may be considered as an intermediate node, and we can consider the rod as half of a rod of double the length, free or fixed at both ends in which the number of nodes is $2n-1$. Replacing l by $2l$ and n by $2n-1$, we then have

$$d = \frac{4l}{4n-3}, \qquad s' = \frac{5l}{4n-3}, \qquad s = \frac{1.3216\, l}{4n-3}, \qquad (281)$$

of which the last two apply only to the case where one of the ends is free.

6°. If the supported ends be regarded as intermediate nodes we have

$$d = \frac{l}{n-1}. \qquad (282)$$

235. ***Harmonic Vibrations of Elastic Rods.*** When the vibrating parts are known, the harmonics of the rod are easily determined, and considering the fixed extremities as nodes, the formulæ of Lissajous above given become general for the six cases. In the first three cases the sounds resulting are the same for the same number of nodes, whatever be the condition of the extremity, whether fixed or free. The numbers of vibrations are as $3^2, 5^2, 7^2, \ldots (2n-1)^2$, when there are 2, 3, 4, ... n nodes. In the 4° and 5° cases where one of the extremities is supported, the vibrational numbers are as $5^2, 9^2, 13^2, \ldots (4n-3)^2$; and in the 6° case the numbers are $1^2, 2^2, 3^2, \ldots (n-1)^2$, n being the number of nodes. Comparing in all these cases the vibrational numbers for two nodes, we have

$$9 : \frac{25}{4} : 4, \qquad \text{or} \qquad \frac{9}{4} : \frac{25}{16} : 1;$$

and therefore generally

$$\frac{(2n-1)^2}{4} : \frac{(4n-3)^2}{16} : (n-1)^2.$$

For d we have

$$\frac{2l}{2n-1}, \qquad \frac{4l}{4n-3}, \qquad \frac{l}{n-1}.$$

Substituting the value of n taken from the latter in the former, we have

$$N = \frac{l^2}{d^2}. \qquad (283)$$

If $d = l$, which corresponds to a rod supported at both ends and yielding its fundamental sound, we have $N = 1$. We therefore conclude that when a rod gives a harmonic, the parts comprised between the nodes vibrate as rods whose extremities are supported and whose length is the distance between the nodes, and that the vibrational number is inversely as the square of this length. This conclusion is inapplicable to the first nodes, because they are more or less influenced by the extremities.

236. *Tuning Forks.* A tuning fork may be regarded as a rod or bar free at both ends. Experiment shows that in proportion as a bar free at both ends is bent or curved the extreme nodes approach each other. Thus, in the figure the bar ab if supported at the points 1, 2, one fourth the length of the bar from the extremes, will when vibrated transversely develop nodes at these points. In the forms $a'b'$, $a''b''$, $a'''b'''$, the length remaining unchanged, the nodes approach each other as indicated in the figure. The laws which govern the vibration of a fork whose section is rectangular have been experimentally found to be; 1°, that the vibrational number is independent of the width; 2°, proportional to the thickness; 3°, inversely proportional to the square of the length increased slightly. The length is taken as equal to the projection of the prongs on the medial line of the fork. For a fork of rectangular cross-section we have from the experiments of Mercadier.

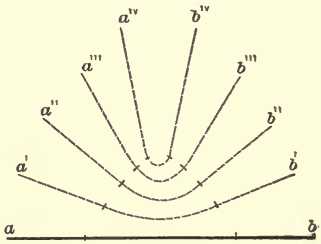


Figure 32.

$$N = \kappa \frac{t}{2l^2(1.012)^2} \tag{284}$$

in which N is the vibrational number, t the thickness, and l the length; κ is a constant which for steel is found to be 818270. When the fork yields its fundamental note its method of division is shown in the figure. The overtones of a fork correspond to vibrational numbers which are to each other, beginning with the first; as, $3^2 : 5^2 : 7^2$: etc. The vibrational number of the first overtone is about $\frac{25}{4}$

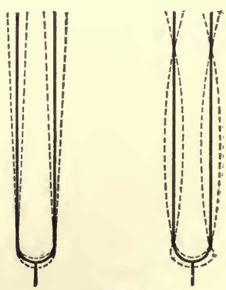


Figure 33.

that of the fundamental. Helmholtz found by experimenting on many forks, that it varied from 5.8 to 6.6 that of the fundamental. These overtones are so high, that they are generally of short duration, and they are also inharmonic with the prime. Tuning forks are generally mounted on their resonant boxes, by which arrangement the prime tone of the fork is greatly reinforced to the disadvantage of the overtones. The duration of the vibration of a fork although theoretically constant, is found to increase slightly with an increase of amplitude and temperature, thus slightly lowering the pitch. This is, however, not appreciable to the hearing, but can be detected by any of the graphical methods for determining the number of vibrations in a given period. It is a matter of importance in determining the initial velocity of projectiles, by means of the Schultz chronoscope or other devices, where the vibrations of a tuning fork enter into the calculation, to limit the amplitude and to take note of the temperature, in order to obtain uniform and reliable results. When the amplitude does not surpass 3 or 4 mm. and the temperature varies but little beyond the ordinary atmospheric temperature, the vibrational number may be taken as constant within .0001 of its value.

237. ***Vibration of Plates.*** Plates are rigid bodies, generally of metal or glass, whose length and breadth are very great compared with their thickness. To put them in vibration, one or more points are fixed and a violin bow is drawn across an edge. The circumstances of vibration are exhibited by sprinkling fine sand over the surface and examining the nodal lines formed by the sand which seeks that part of the plate which is at rest. The parts of the plate separated by a nodal line, evidently vibrate in opposite directions, and therefore for permanent figures the number of vibrating parts must be even. When the plate yields its fundamental tone the resulting figure is the simplest that can be formed, and as the plate separates into a greater number of vibrating parts, the figures become more complex. Chladni has given to these figures the name of *Acoustic figures.* As yet, from the inherent difficulties of the problem, the mathematical laws have not been deduced, but experiment has assigned the following as the laws of vibrating plates, viz.; 1°, the vibrational numbers of plates of the same form and of the same material are inversely as the squares of

the homologous dimensions ; 2°, and are proportional to the thickness. Hence we have

$$n : n' :: \frac{t}{l^2} : \frac{t'}{l'^2}.$$

If a rectangular plate be so constructed that a system of nodal lines parallel to the length be formed by a sound, which gives another system of nodal lines parallel to the breadth, when it is vibrated in these two ways, then if at any of the middle points of the ventral segments it be vibrated so as to produce the same sound, these two systems will simultaneously exist and the acoustic figure will result from the combination of these two systems. The figure illustrates five such plates where the numbers of the nodal lines are in the ratios of 2:3, 2:4, 3:4, 3:5, 4:5. Other combinations illustrative of the vibrations of plates are reserved for the lectures.

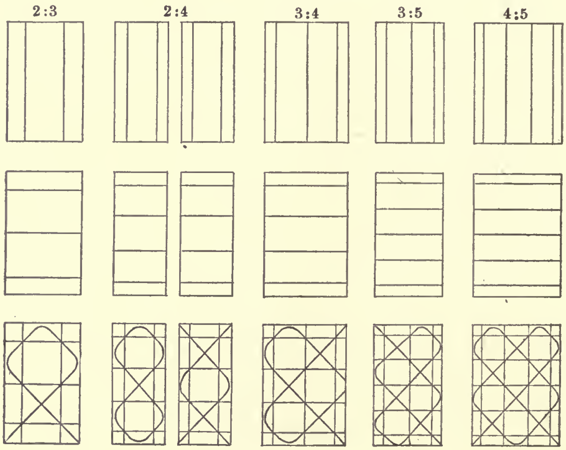


Figure 34.

238. ***Vibration of Membranes.*** When a stretched membrane is near a sounding body, the air transmits to it the vibratory motion. It can respond, however, only to certain sounds depending on its tension, and thus enter into synchronous vibration. This fact is made evident by the acoustic pendulum, or by the nodal lines formed by sand sprinkled upon it, as in the case of the

vibration of plates. The frames upon which the membranes are stretched are generally square or circular. Experiment has confirmed the following deductions of Poisson and Lamé, with respect to the vibrations of square membranes, viz.:

1°. Membranes respond only to certain sounds, separated by determinate intervals.

2°. To each sound a system of nodal lines corresponds, parallel to the sides of the membrane, and whose numbers are represented by n and n'.

3°. The nodal lines which correspond to the same sound form a system of figures, such that we can pass from one to the other by continuous changes in varying the mode of disturbance, without changing the sound; but we can never pass in a continuous manner from the lines of one sound to those of another.

Circular membranes can only give nodal lines along the diameters or circumferences, either separate or combined, depending on the method of vibration and on the point or points of enforced rest.

239. Because of the limited time allotted to this part of the course, many subjects of importance are necessarily omitted in the text. Among these are,

1°. The theory of beats, and resultant sounds.

2°. The phenomena of interference, whose consequences, however, are readily derived from the discussion in Arts. 65–68.

3°. The graphical and optical methods of the study of sonorous vibrations, and that by sensitive and manometric flames.

4°. The phenomena of vibrations of air columns in organ-pipes, of elastic rods, of plates and membranes, with the applications of the latter in the phonograph, phonautograph, and telephone.

By means, however, of a very complete acoustical apparatus, mainly from the workshop of Koenig, the celebrated physicist of Paris, the omitted parts, as well as those treated of above, are illustrated in the lectures, which largely supplement and complete the study of the text.

240. The nature and essential principles of undulatory motion, as illustrated by sonorous vibrations, have received sufficient attention to enable the student to prosecute understandingly the study of similar principles connected with light in the analogous subject of optics.

PART III.

OPTICS.

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241. Light is the agent by which the existence of bodies is made known to us through the sense of sight.

That branch of physical science which treats of the properties of light and the laws of its transmission is called ***Optics.***

242. It is divided into two parts:

1°. ***Geometrical Optics,*** which embraces all the phenomena relating to the propagation of rays, based on certain experimental laws, and which is entirely independent of any theory as to the nature of the luminous agent.

Experiments in Geometrical Optics, however carefully made, can never accurately prove the laws of light propagation, but serve merely to establish a certain degree of probability of their truth, and which, when applied to other phenomena of the same nature, strengthen this probability in proportion as the application is more extended.

2°. ***Physical Optics,*** which is based on the theory of undulations, and seeks to explain by this theory the nature of light, and of all the phenomena arising from the action of rays on each other.

243. That light is not a material substance, but is merely a *process* going on in some medium, is proved by the phenomena of interference, in which results of various magnitudes occur, from less to greater, or the reverse, depending upon the manner in which the interference takes place, even when the combining magnitudes are themselves constant in value.

244. The undulatory theory asserts that light is due to the transmission of energy from luminous bodies to the finely-divided parts of the optic nerve, spread over the interior concave surface of the eye. This energy is conveyed by the optic nerve to the brain, and there transformed into the sensation of sight.

The transmission of the energy is accomplished by undulatory motion in a medium called the *luminiferous ether*. There is no direct proof of the actual existence of the ether, and its assumption can only be regarded as an extremely probable hypothesis, supported by nearly all the known phenomena of light, and directly contradicted by none.

Within the present century, its reality has been almost universally accepted, and as a consequence the undulatory theory has entirely supplanted the rival hypothesis of the materiality of light molecules, known as the emission theory, which had, however, held its ground for many years.

245. The accepted properties of the luminiferous ether have resulted from theoretical considerations, modified from time to time by deductions from experimental observations, and while there are several imperfections yet to be removed, nevertheless the strong array of unquestioned facts, both observed and predicted, has established these properties as a satisfactory foundation upon which modern physical optics is now constructed.

The luminiferous ether is considered to be a material substance of a more rare and subtile nature than the ordinary matter affecting the senses, and to exist not only within these bodies, but throughout space. It has great elasticity, and is capable therefore of transmitting its particular energy over vast distances, with great velocity and with inappreciable loss. That this energy is not transmitted instantaneously has been proved by direct experiment, and concluded from several astronomical observations.

246. That light is propagated in right lines from the source is a fact of observation and experiment. This statement, however, while absolutely true, is subject to modification when taken in the ordinary sense of the language. Thus, we have seen that while sound is propagated in right lines from its source, it is capable of spreading around an obstacle, so that sound can be heard out of the direct line of the source ; so, in a less degree, we can see around an obstacle, as will be shown in the discussion of the diffraction of light.

The acoustic shadow, however, is as much less marked than the optical shadow as the wave lengths of sound are greater than the wave lengths of light. But for the explanation of the principles of geometrical optics it is unnecessary to consider this refinement.

247. Bodies are called *self-luminous* when they are themselves the sources of light, and rays proceed directly from them. They are visible because of their emanating rays. Other bodies are called *non-luminous*, and become visible because rays from luminous bodies are reflected from their surfaces.

A luminous point, or origin of light, is a very small portion cf a luminous surface. When light emanates from a luminous point, we consider it made up of *rays* of light, each of which is the smallest portion of light which can be transmitted. The ray is the right line along which the undulation is propagated, and is practically a mere conception, indicating direction.

A collection of parallel, diverging, or converging luminous rays is called a *beam* of light, and sometimes a *pencil* of light, the latter name being generally applied to the last two cases.

The *axis* of a beam is the geometrical axis of the cylinder or cone of rays ; when the axis is normal to the deviating surface, the beam is *direct*, and when inclined to it, *oblique*.

248. When a beam of light is incident upon any surface, it is generally separated into three portions, viz., a part is scattered or diffused over the surface, by which the surface becomes visible, a second part is reflected, and the remainder is refracted.

The proportion of the several parts depends on the polish of the surface, the angle of incidence, and the nature of the medium. A perfectly polished surface would be invisible, and the incident beam would be separated into a reflected and refracted beam alone; of course, such a polish is not practicable. Light regularly reflected has its intensity increased with the degree of polish, while the intensity of irregularly reflected light is similarly diminished. The intensity of regularly reflected light from the surface of water is, at the incidences of

	0°,	40°,	60°,	80°,	$89\frac{1}{2}$°,
about	1.8%,	2.2%,	6.5%,	33%,	72%.

At normal incidence, water, glass, and mercury reflect 1.8%, 2.5%, and $66\frac{2}{3}$%, respectively. The differences at small angles of incidence are more marked than at greater angles, since while both water and mercury reflect the same at $89\frac{1}{2}°$, the former reflects but $\frac{1}{36}$ as much as the latter at normal incidence.

249. A *medium* is any substance which permits the passage of light through it.

Since the luminiferous ether is supposed to pervade all matter, it might be inferred that all bodies could be classed under the head of media for light. Gold, although one of the most dense of substances, does permit the passage of light, when beaten into a very thin leaf; and no doubt if other opaque bodies possessed an equal malleability, the same property would belong to them.

But owing to internal reflection and consequent interference, it is assumed that an inappreciable quantity of light, if any, passes through very small thicknesses of *opaque* bodies. Glass, air, water, and all other matter which permit the passage of light freely, are said to be transparent. *Translucency* is a term applied to such bodies as permit the passage of diffused light; thus, ground glass and flint are translucent, while clear glass and quartz crystal are transparent.

250. Since light is assumed to result from undulatory motion in the luminiferous ether, all the consequences deduced in the discussion of the properties of this kind of motion in Part I are at once applicable to the phenomena of light.

251. ***Shadows and Shade.*** From each luminous point considered as an origin of disturbance, undulations proceed along right lines in all directions from this origin. Therefore, whenever they meet an opaque body, this undulation will be deviated from its original direction, and the effect of light will be wanting along this direction prolonged.

The absence of this effect is called the *shadow* of the point of the opaque body.

The line of the surface of the opaque body, along which rays drawn from the luminous point are tangent, is called the *line of shade.* Since each point of the luminous surface is an origin of light, we see that in all actual cases the shadow of an opaque body must be indistinct near its boundary, and gradually merge into the illuminated surface surrounding the shadow, whenever the luminous source is of an appreciable area. This modified portion of the shadow is due to the overlapping of the cones of rays proceeding from each luminous point, and is called the *penumbra.* It is limited by the space between the two cones, whose elements are tangent to the luminous surface and the opaque body, one having its vertex between the two, and the other its vertex on the further side of either one of the surfaces. The softness of shadows in general is due to the finite extent of luminous surfaces.

252. Every point of the luminous source emitting rays in all directions, each will carry an image of its luminous point.

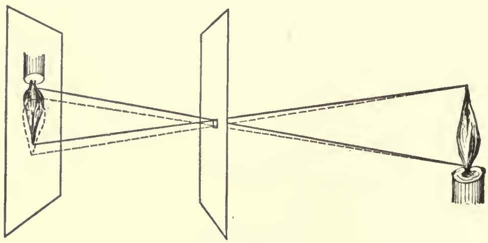

Figure 35.

Thus, if a lighted candle be placed in front of a small aperture of a darkened chamber, the aperture will permit the passage of a limited number of the rays from every point of the candle, each ray, however, carrying an image of its radiant. The image, as shown in Figure 35, will be inverted.

If another aperture be made near the first, a second image of the candle will be formed, overlapping the first, and, while the luminosity will be increased, the image will lose distinctness, because of this overlapping. The diffused light of a room during the day is due to the overlapping images of external objects, caused by rays proceeding from each of them, thus making their individual images indistinct. A small aperture in a darkened room will permit the formation of an inverted image of the external scenery upon a screen placed within the room near the aperture.

253. ***Photometry.*** The eye possesses the property of distinguishing color and intensity.

In determining variations of intensity, the judgment is only approximate when the colors are the same, and the difficulty of this appreciation is increased when the colors differ. Equality of intensity can readily be determined by the eye, while it is not possible to ascertain the numerical ratio of different intensities by direct observation.

Photometry has for its object the measurement and comparison of the intensities of different lights.

254. The principle of all photometric methods is to arrive at this comparison, by the appreciation of the equality of illumination of two near surfaces, physically identical. In assuming the distance of the luminous source from the illuminated surface to be great in comparison with the dimensions of the surface, and remembering that the intensity of the light is due to the molecular kinetic energy, we readily see, if there be no absorption of this energy during transmission through the intervening media,

1°. That the intensity of the illumination on the unit area of any surface, taken normal to the direction of propagation, at a distance d from the luminous source, varies as $\frac{1}{d^2}$.

2°. That if I represent the intensity of any given light, and if it be supposed to illuminate uniformly any area A, the intensity on a unit of area varies as $\frac{1}{A}$.

3°. That the quantity of light emanating from any luminous element, and hence the intensity of illumination on the unit area, is proportional to the cosine of the angle made by the normal to the element with the direction considered, and hence varies as the cosine of the inclination, or $\cos i$.

4°. That if the area on which the light falls is inclined to the direct line of propagation, the illumination on the unit area is proportional to the cosine of the angle made by this line and the normal to the surface, or to $\cos i'$.

5°. That the illumination on the unit area will vary with the *intrinsic brightness* of the source. The intensity of the illumination on the unit area, parallel to the source, at the distance unity, may be taken as the measure of the intrinsic brightness.

255. Let S and S' be the projections of the luminous and the illuminated surfaces, respectively, on a plane normal to the direction of the luminous rays; B the intrinsic brightness of the source ; d the distance apart of the two surfaces, and I the intensity of the illumination ; then, from the above principles, we have

$$I = B\frac{SS'}{d^2}. \tag{285}$$

Making $S' = 1$, and calling $I_,$ the total brilliancy of the source at the distance d, we have

$$I_, = B\frac{S}{d^2}. \tag{286}$$

$\frac{S}{d^2}$ is the apparent area of the source seen from the illuminated surface, and making this equal to unity, we have

$$I_{,,} = B. \tag{287}$$

Therefore the intrinsic brightness of the source is the total brilliancy of the apparent unit of area of the luminous surface at the distance 1.

The general method of comparison of the intrinsic brightness of two sources consists in permitting the rays from each source to fall, nearly normal, upon adjacent portions of the same surface; then to increase the distance of the stronger light, until the eye judges the illumination to be equal. We then have

$$\frac{BS}{d^2} = \frac{B_,S_,}{d_,^2}; \tag{288}$$

from which, by substituting the known values of d, $d_,$, S and $S_,$, the ratio of $B_,$ to B can be determined.

256. The *apparent* intrinsic brightness of an object is equal to the quantity of light received from it by the eye, divided by the area of the picture on the retina. Therefore, since the apparent illumination of the object is $B\frac{S}{d^2}$, and the area of the retinal picture is $\frac{S}{d^2}$, the apparent intrinsic brightness will vary with the real intrinsic brightness B, and the object will appear equally bright at all distances.

This result is deduced under the supposition that no light from the object is absorbed by the medium through which it passes, and is therefore only an approximation.

257. *Velocity of Light.* In 1675, the Danish astronomer, Rœmer, noticed certain discrepancies with regard to the observed times of the eclipses of Jupiter's satellites, which he correctly attributed to the finite velocity of light.

To show this, let S be the sun, EE′ the earth's orbit, JJ′ the orbit of Jupiter, and *ss*′ the orbit of Jupiter's inner satellite.

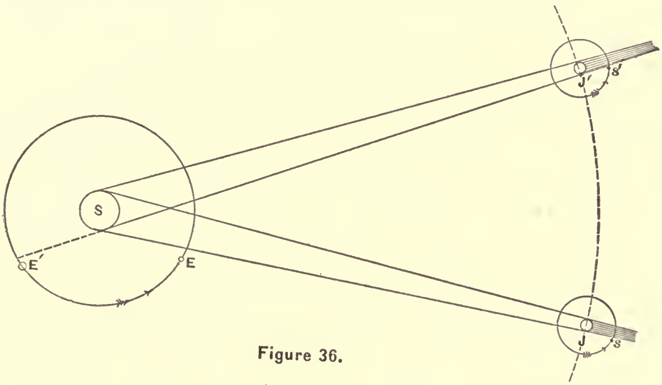

Figure 36.

The planets and satellites shine by the reflected light of the sun, and therefore cast shadows, whose axes are on the right lines joining their centres with the centre of the sun. Because of the position of the orbit of the satellite with respect to the plane of Jupiter's orbit, the satellite enters Jupiter's shadow at every revolution, and is eclipsed. If light traversed space instantaneously, its entrance into and exit out of the shadow might be noted at the exact instants at which these phenomena occurred, independently of the relative positions of the earth and Jupiter.

But when Jupiter is near opposition, as at J, the interval between two successive disappearances of the satellite in entering, or between two successive reappearances on emerging from the shadow is found to be about 42 hr. 30 min. The periodic time of Jupiter being about 11 yr. 10 mo., he advances but a short distance, as to J′, while the earth moves to E′ near conjunction.

Their distance is now increased by very nearly that of the diameter of the earth's orbit, and the times of apparent immersion of the satellite are delayed beyond the computed times by about 16 min. 26 sec. Since the periodic time of the satellite is constant, Rœmer therefore concluded that light required 16 min. 26 sec. to traverse this diameter.

If this diameter were accurately known, and the exact instant of the eclipse could be noted, a very nearly exact measure for the velocity of light could be computed. The reduction of more than a thousand eclipses of Jupiter's satellites, by Delambre, gave 473.2 mean solar seconds for the time of travel, which corresponds to a solar parallax of 8.878″, and to a velocity of 298,793 kilometres per second.

258. ***By the Aberration of Light.*** Bradley, in 1728, accounted for the aberration of the fixed stars by assuming that the velocity of the earth's orbital motion had an appreciable ratio to the velocity of light. By assuming an ideal star at the pole of the ecliptic, the value of the constant of aberration, according to his determination, is 20.25″, which corresponds to a solar parallax of 8.881″. According to W. Struve, this constant should be 20.445″, decreasing the parallax to 8.797″, and corresponding to a velocity of 296,067 kilometres per second.

The principle on which this method is based is given in the text on Astronomy.

259. ***By Actual Measurement.*** Owing to the great velocity of light, it is not possible to measure directly the very small interval of time required for light to traverse any terrestrial distance. But Fizeau, Foucault, Wheatstone, Cornu, and more recently Michelson, have succeeded in obtaining its value within very near limits. The essential principle of the experiment by Fizeau consists in causing a toothed wheel to revolve with great, but uniform velocity, in a plane perpendicular to the track of a small parallel beam of light. The toothed wheel in its rotation alternately permits and obstructs the passage of the beam, according as an interval or a tooth is interposed in its track. The beam of light, after traversing the distance determined upon, is reflected by a small mirror, and may or may not be intercepted on its return, depending on the ratio of the velocity of rotation of the wheel and the velocity of light. Should the velocity of rotation be such that the returning beam passes through the next interval, the circumference of the wheel would have moved through an angle equal to that subtended by a tooth and an interval, while the light has traversed double the distance from the wheel to the reflector.

When the angular velocity of the wheel is doubled, the light passes through the second interval, and so on. The value for the velocity of light determined by this method is 315,364 kilometres. Cornu has recently made use of the same method, but with a very much improved apparatus, and has found, as the mean of 504 experiments, the value of 300,400 kilometres for the velocity of light in vacuo, with a probable error of less than .001.

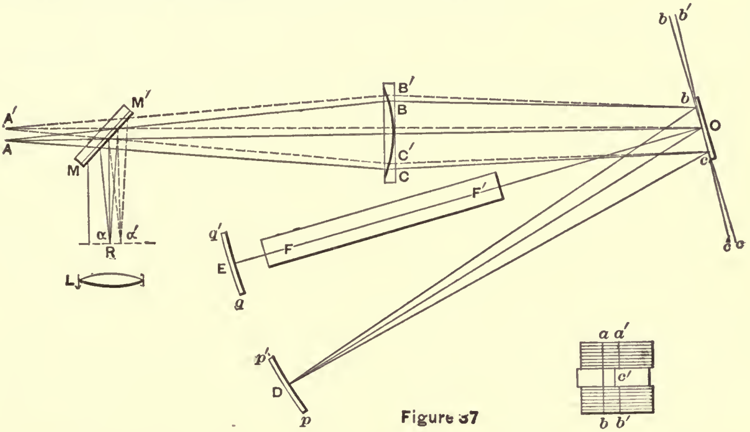

Figure 37

260. Foucault's method is a modification of the preceding. Let A, in Figure 37, be a luminous line, BC a lens whose focal length for the position A is Bb + bD, bOc a revolving mirror, D and E circular mirrors whose centre is at O, MM′ a glass plate, R a reticle, and L an eye lens to view the image of A. Now if the mirror O is at rest, the path of a ray from A, passing through the lens BC and reflected from O, is ABbD; returning by reflection from D, its path is DcCA. A part of its light is reflected from the first surface of MM′, and the image of A is seen coincident with its object at α. If now the mirror O is put into sufficiently rapid rotation, the returning ray meets it at b'Oc', and the ray is reflected along cC′A′, and its image is seen at α'. The angle bOb' is known from the velocity of rotation, the distance OD is given, and the displacement $\alpha\alpha'$ is measured by a micrometer.

These data serve to measure the velocity of light in terms of the angular velocity of O. By the addition of a tube filled with water at FF′, the velocity of light in water was found and shown to be less than that in air.

In the diagram annexed to the figure, *ab* is the position of the image when O is at rest, *c*′ when O has a determinate velocity, and *a*′*b*′ the corresponding position of the image after the ray has traversed the water. The result of this determination is 298,187 kilometres for the velocity of light, corresponding to a solar parallax of 8.86″. Michelson, by an ingenious modification of the method of Foucault, by which he separated his mirrors 2000 feet, and caused one of them to revolve 257 times per second, obtained a deflection of his image exceeding 133 millimetres, and thus obtained results which are claimed to be exact to within one ten-thousandth, due to this element of deflection.

As the mean of 1000 observations, he has determined 299,930 kilometres per second for the velocity of light in vacuo.

A new investigation of this important constant, under the direction of Prof. Newcomb, is now in progress, and which, when completed, will undoubtedly be as close an approximation to the true value as the present state of experimental science can furnish.

261. Assuming that light is due to the transversal vibrations of the luminiferous ether, we see, Eq. (119), that in isotropic media the velocity of light depends on the coefficients a, b, c, etc., which are functions of the elasticity and density of the medium.

In homogeneous light, or that in which λ is constant, V will therefore vary when light passes from one medium into another. The conclusions derived by supposing a variation in λ, the medium remaining the same, will be considered under the dispersion of light.

GEOMETRICAL OPTICS.

262. In geometrical optics it is only necessary to take account of the variation of the velocity due to a change in the elasticity and density of the ether, in passing from one isotropic medium into another. Hence, we consider homogeneous light alone in the discussions which follow. These changes are given by the formula

$$\sin\phi = \mu \sin\phi' = \frac{V}{V'}\sin\phi'. \qquad (289)$$

The ratio μ is called the *index of refraction;* it is the ratio of the velocity of light propagation in the two media, and is called the *absolute* index when the medium from which it passes is the ether. When any two other velocities are compared, the ratio is called the *relative* index; the relative index is then only the ratio of the two absolute indices. When reflection is considered as a particular case of refraction, μ is always taken as -1.

263. A *radiant* is a point from which the rays proceed; it is said to be *real* when the beam is parallel or diverging, and *virtual* when converging. A *focus* is the point in which the rays meet after deviation, or in which they would meet if prolonged in either direction; in the former case the focus is real, and in the latter, if the point of meeting is found by prolonging the rays backward, it is virtual. A radiant and its focus are the centres of curvature of the undeviated and deviated pencils, respectively. In the following discussions, distances estimated in the direction of wave propagation, from any origin whatever, are taken as *negative,* and in the contrary direction as *positive.*

264. ***Deviation of Light by Plane Surfaces.*** Let us suppose the incident medium to be any whatever, as air, and that the ray enters any other medium, as glass, whose surface is plane, Then, Figure 38, we have, for the first refraction,

$$\sin \phi = \mu \sin \phi', \tag{290}$$

in which μ is the relative index of air referred to glass; and for the first reflection we have

$$\sin \phi = -\sin \phi. \tag{291}$$

The angle ϕ' is less than ϕ, because $\mu = \frac{V}{V_{\prime}}$ is greater than unity, since the velocity of wave propagation of light in air is found by experiment to be greater than that in glass. Should the velocity in the medium of intromittance be greater than that in the medium of incidence, μ would be less than unity and ϕ' would be greater than ϕ. The re-

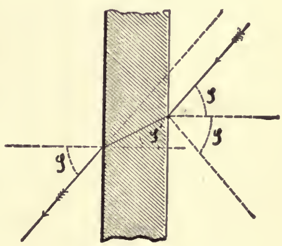

Figure 38.